A Combinatorial Proof for the Circular Chromatic Number of Kneser Graphs

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Abstract

Chen [4] confirmed the Johnson-Holroyd-Stahl conjecture that the circular chromatic number of a Kneser graph is equal to its chromatic number. A shorter proof of this result was given by Chang, Liu, and Zhu [3]. Both proofs were based on Fan's lemma [5] in algebraic topology. In this article we give a further simplified proof of this result. Moreover, by specializing a constructive proof of Fan's lemma by Prescott and Su [19], our proof is self-contained and combinatorial.

1 Introduction

Let G be a graph and t a positive integer. A *proper* t-coloring of G is a mapping that assigns to each vertex a color from a set of t colors such that adjacent vertices must receive different colors. The *chromatic number* of G denoted as $\chi(G)$ is the smallest t of such a coloring admitted by G. Let $n \geqslant 2k$ be positive integers. The *Kneser graph* $\mathrm{KG}(n,k)$ has the vertex set $\binom{[n]}{k}$ of all k-subsets of $[n] = \{1, 2, 3, \ldots, n\}$, where two

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vertices A and B are adjacent if $A \cap B = \emptyset$. Figure 1 shows an example of $\mathrm{KG}(5,2)$ with a proper 3-coloring.

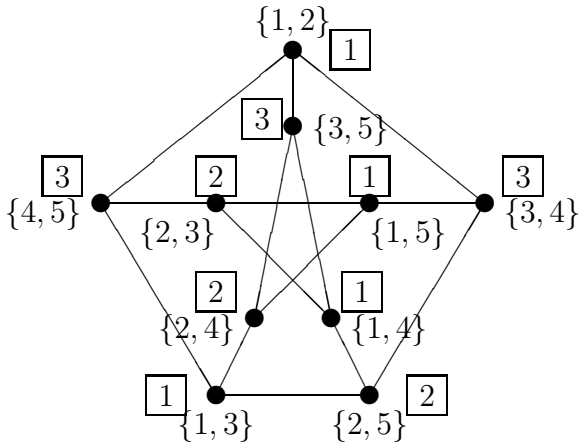


Figure 1: A proper 3-coloring of $\mathrm{KG}(5,2)$ (also known as Petersen graph).

Lovász [15] in 1978 confirmed the Kneser conjecture [11] that the chromatic number of $\mathrm{KG}(n,k)$ is equal to $n-2k+2$. Lovász's proof applied topological methods to a combinatorial problem. Since then, algebraic topology has became an important tool in combinatorics. In particular, various alternative proofs (cf. [2, 7, 17]) and generalizations (cf. [1, 12, 13, 16, 20, 21]) of the Lovász-Kneser theorem have been developed. Most of these proofs utilized methods or results in algebraic topology, mainly the Borsuk-Ulam theorem and its extensions.

Theorem 1. (Lovász-Kneser Theorem [15]) *For any $n \geqslant 2k$,*

$$\chi(\mathrm{KG}(n,k)) = n-2k+2.$$

In 2004, Matoušek [17] gave a self-contained combinatorial proof for the Lovász-Kneser Theorem by utilizing the Tucker Lemma [23] together with a specialized constructive proof for the Tucker Lemma by Freund and Todd [6]. Later on, Ziegler [27] gave combinatorial proofs for various generalizations of the Lovász-Kneser Theorem.

For positive integers $p \geqslant 2q$, a (p,q)-*coloring* for a graph G is a mapping $f : V(G) \to \{0,1,2,\ldots,p-1\}$ such that $|f(u)-f(v)|_p \geqslant q$ holds for adjacent vertices u and v, where $|x|_p = \min\{|x|, p-|x|\}$. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is the infimum p/q of a (p,q)-coloring admitted by G. It is known (cf. [24, 25]) that $\chi_c(G)$ is rational if G is finite, and the following hold for every graph G:

$$\chi(G)-1 < \chi_c(G) \leqslant \chi(G). \tag{1.1}$$

Thus the circular chromatic number is a refinement of the chromatic number for a graph. The circular chromatic number reveals more information about the structure of a graph than the chromatic number does. Families of graphs for which the equality $\chi_c(G) = \chi(G)$ holds possess special structure properties and they have been broadly studied (cf. [24, 25]). Kneser graphs turned out to be an example among those widely studied families of graphs.

Johnson, Holroyd, and Stahl [10] conjectured that $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. This conjecture has received much attention. The cases for $k = 2$, and $n = 2k + 2$ was confirmed in [10]. By a combinatorial method, Hajiabolhassan and Zhu [9] proved that for a fixed k, the conjecture holds for sufficiently large n. Using topological approaches, Meunier [18] and Simonyi and Tardos [22] confirmed independently the case when n is even. Indeed, all these results were proved true [9, 14, 18, 22] for the *Schrijver graph* $\mathrm{SG}(n,k)$, a subgraph of $\mathrm{KG}(n,k)$ induced by the k-subsets of $[n]$ that do not contain adjacent numbers modulo n. On the other hand, it was shown by Simonyi and Tardos [22] that for any $\epsilon > 0$, there exists $\delta > 0$ such that if n is odd and $n - 2k \leqslant \delta k$, then $\chi_c(\mathrm{SG}(n,k)) \leqslant \chi(\mathrm{SG}(n,k)) - 1 + \epsilon$. Hence the Johnson-Holroyd-Stahl conjecture cannot be extended to Schrijver graphs.

In 2011, Chen [4] confirmed the Johnson-Holroyd-Stahl conjecture. A simplified proof for this result was given by Chang, Liu, and Zhu [3]. At the center of both proofs is the following:

Lemma 2. (Alternative Kneser Coloring Lemma [4, 3]) *Suppose* $c : \binom{[n]}{k} \to [n - 2k + 2]$ *is a proper coloring of* $\mathrm{KG}(n,k)$. *Then* $[n]$ *can be partitioned into three subsets,* $[n] = S \cup T \cup \{a_1, a_2, \ldots, a_{n-2k+2}\}$, *where* $|S| = |T| = k - 1$, *and* $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$ *for* $i = 1, 2, \ldots, n - 2k + 2$.

Let c be a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$. The Lovász-Kneser Theorem is equivalent to saying that every color class in c is non-empty. Lemma 2 strengthens this result by revealing the exquisite structure of a Kneser graph induced by an optimal coloring. For instance, the proper 3-coloring in Figure 1 has $a_i = i$ for $i = 1, 2, 3$, $S = \{4\}$, and $T = \{5\}$. By Lemma 2, the subgraph of $\mathrm{KG}(n,k)$ induced by the vertices $S \cup \{a_i\}$ and $T \cup \{a_i\}$, $1 \leqslant i \leqslant n - 2k + 2$, is a fully colored (i.e. uses all colors) complete bipartite graph $K_{n-2k+2,n-2k+2}$ minus a perfect matching. Moreover, the closed neighborhood for each vertex in this subgraph is fully colored.

It is known (cf. [8]) that this fact easily implies that $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. For completeness, we include a proof of this implication.

Theorem 3. [4, 3] *For positive integers* $n \geqslant 2k$, $\chi_c(\mathrm{KG}(n,k)) = n - 2k + 2$.

Proof. Assume to the contrary that $\chi_c(\mathrm{KG}(n,k)) = p/q$ where $\gcd(p,q) = 1$ and $q \geqslant 2$. Let $d = n - 2k + 2$. By (1.1), it must be $(d-1)q < p < dq$. Let f be a (p,q)-coloring for $\mathrm{KG}(n,k)$. The function c defined on $\binom{[n]}{k}$ by $c(v) = \lfloor f(v)/q \rfloor$ is a proper coloring of $\mathrm{KG}(n,k)$ using colors in $\{0, 1, 2, \ldots, d-1\}$.

By Lemma 2, there is a partition $[n] = S \cup T \cup \{a_0, a_1, \ldots, a_{n-2k+1}\}$ such that $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$ for $0 \leqslant i \leqslant n-2k+1$. Denote $S_i = S \cup \{a_i\}$ and $T_i = T \cup \{a_i\}$ for $i = 0, 1, \ldots, d-1$. By the definition of c, we obtain

$$iq \leqslant f(S_i), f(T_i) < \min\{(i+1)q, p\}, \text{ for } i = 0, 1, 2, \ldots, d-1.$$

Assume $f(S_0) \geqslant f(T_0)$ (the other case can be proved similarly). Then $f(T_1) \geqslant f(S_0) + q$ and $f(S_2) \geqslant f(T_1) + q$, implying $f(S_2) \geqslant f(S_0) + 2q$. Continue this process until the last term. If d is even, we obtain $f(T_{d-1}) \geqslant f(S_0) + (d-1)q$. Because S_0 and T_{d-1} are adjacent, so $|f(S_0) - f(T_{d-1})|_p \geqslant q$. This implies that $p - f(T_{d-1}) + f(S_0) \geqslant q$. Hence, $p \geqslant dq$, a contradiction.

If d is odd, we obtain $f(S_{d-1}) \geqslant f(S_0) + (d-1)q$. Because T_0 and S_{d-1} are adjacent, so $|f(T_0) - f(S_{d-1})|_p \geqslant q$. This implies that $p - f(S_{d-1}) + f(T_0) \geqslant q$. Since $f(S_0) \geqslant f(T_0)$, so $p \geqslant dq$, a contradiction. Thus Theorem 3 follows. □

Both proofs of Lemma 2 in [4, 3] utilized Fan's lemma [5] applied to the boundary of the barycentric subdivision of n-cubes. The aim of this article is to present a proof for Lemma 2, which on one hand is a self-contained combinatorial proof, and on the other hand, further simplifies the proof presented in [3].

Our proof of Lemma 2, presented in the next two sections, is established by modifying a constructive proof for Fan's lemma given by Prescott and Su [19] to the desired special case, together with the labeling scheme used in [3]. The proof for the labeling scheme is further simplified and more straightforward than the one in [3]. In addition, our modification of the constructive proof in [19] corrects a minor error occurred in that paper.

2 Labeling of $\{0, 1, -1\}$-vectors

We present a proof of the Fan's lemma [5] applied to the boundary of the first barycentric subdivision of the n-cubes. The proof is by modifying and specializing the constructive proof of Fan's lemma given by Prescott and Su [19].

Let n be a positive integer and $\mathcal{F}^n = \{0, 1, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$ be the family of vectors $A = (a_1, a_2, \ldots, a_n)$, where each $a_i \in \{0, 1, -1\}$, and $a_j \neq 0$ for at least one j. A vector $A \in \mathcal{F}^n$ can also be expressed as $A = (A^+, A^-)$ where $A^+ = \{i : a_i = 1\}$ and $A^- = \{i : a_i = -1\}$. Let $|A| = |A^+| + |A^-|$. Notice that $A^+ \cap A^- = \emptyset$, and $|A| \geqslant 1$. For $A = (A^+, A^-), B = (B^+, B^-) \in \mathcal{F}^n$, we write $A \leqslant B$ if $A^+ \subseteq B^+$ and $A^- \subseteq B^-$. If $A \leqslant B$ but $A \neq B$, then $A < B$.

Let n, m be positive integers. Let λ be an m-labeling (mapping) from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm m\}$. We say λ is *anti-podal* if $\lambda(-X) = -\lambda(X)$ for all $X \in \mathcal{F}^n$. Two vectors $X, Y \in \mathcal{F}^n$ form a *complementary pair* if $X < Y$ and $\lambda(X) + \lambda(Y) = 0$. In the

following, we assume that λ is an anti-podal labeling of $\mathcal{F}^n$ without complementary pairs.

A non-empty subset σ of $\mathcal{F}^n$ is called a *simplex* if the vectors in σ can be ordered as $A_1 < A_2 < \cdots < A_d$. Since $|A_d| \leqslant n$, if σ is a simplex, then $1 \leqslant |\sigma| \leqslant n$. Figure 2 shows an example of $\mathcal{F}^3$.

Topologically, each vector $A \in \mathcal{F}^n$ is a point on the boundary of the n-dimensional cube (with a_i be the ith coordinate of the point), and a simplex σ defined above is the convex hull of the points in σ. Although our proof does not use the topological meaning of this concept, this topological background can be helpful in understanding the arguments.

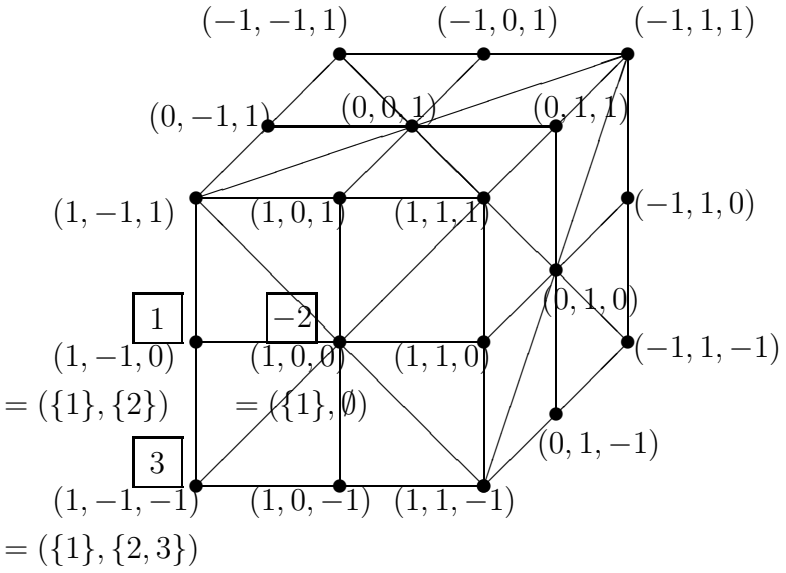


Figure 2: Vertices and points in $\mathcal{F}^3$, where each triangle is a simplex of three vertices. The boxed numbers (labels) show an example of a positive alternating simplex $\sigma : A_1 < A_2 < A_3$, where $A_1 = (1, 0, 0), A_2 = (1, -1, 0), A_3 = (1, -1, -1)$, and $\lambda(\sigma) = \{1, -2, 3\}$.

A simplex $\sigma = A_1 < A_2 < \cdots < A_d$ is *alternating* with respect to λ if the set $\lambda(\sigma) = \{\lambda(A_1), \lambda(A_2), \ldots, \lambda(A_d)\}$ of labels can be expressed either as $\{k_1, -k_2, k_3, \ldots, (-1)^{d-1}k_d\}$ or as $\{-k_1, k_2, -k_3, \ldots, (-1)^d k_d\}$, where $1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant m$. In the former case, $\text{sign}(\sigma) = 1$ and σ is *positive alternating*; in the latter case, $\text{sign}(\sigma) = -1$ and σ is *negative alternating*.

A simplex σ is *almost-alternating* if it is not alternating, but the deletion of some element from σ results in an alternating simplex. Since there are no complementary pairs, every almost-alternating simplex contains exactly two elements such that the deletion of each of them from σ results in an alternating simplex. Moreover, both

resulting alternating simplexes are of the same sign. This common sign is defined as $\text{sign}(\sigma)$.

The *maximum non-zero index* of a simplex, $\sigma = A_1 < \cdots < A_d$, is $\max(\sigma) = \max\{i :$ the i-th term of A_d is non-zero$\}$. Denote $\beta(\sigma)$ as the $(\max(\sigma))$-th term of A_d. An alternating or almost-alternating simplex σ is *agreeable* if $\beta(\sigma) = \text{sign}(\sigma)$.

Lemma 4. [5] *Assume $\lambda : \mathcal{F}^n \to \{\pm 1, \pm 2, \ldots, \pm m\}$ is an anti-podal labeling without complementary pairs. Then there exist an odd number of positive alternating simplexes of size n. Consequently, $m \geqslant n$.*

Figure 3 shows examples of Lemma 4 for $n = m = 2$.

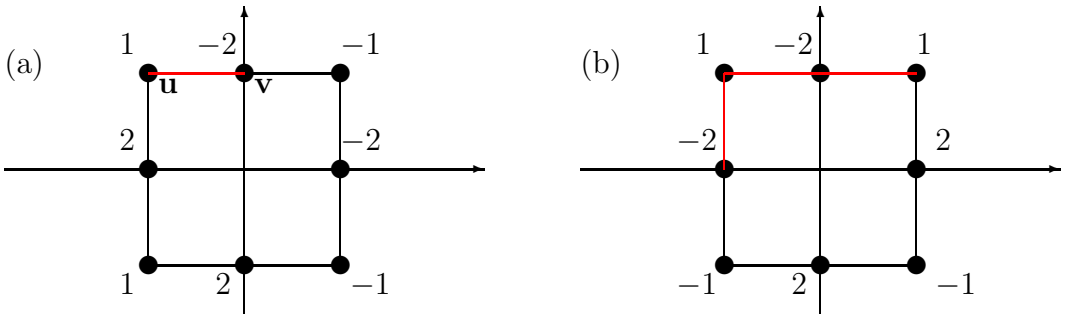


Figure 3: There are 8 vectors (points) in $\mathcal{F}^2$. In each (a) and (b), the numbers on the vectors form an anti-podal 2-labeling without complementary pairs. In (a) there is only one positive alternating simplex of size 2, namely uv, while in (b) there are three such simplexes.

Proof. Define a graph G with the following three types of simplexes σ as vertices.
Type I: $\max(\sigma) = |\sigma| + 1$, and σ is agreeable alternating.
Type II: $\max(\sigma) = |\sigma|$, and σ is agreeable almost-alternating.
Type III: $\max(\sigma) = |\sigma|$, and σ is alternating.

Two vertices σ and τ are adjacent in G if all the following conditions are satisfied:
(1) $\sigma \subset \tau$, $|\sigma| = |\tau| - 1$,
(2) σ is alternating,
(3) $\beta(\tau) = \text{sign}(\sigma)$, and
(4) $\max(\tau) = |\tau|$.

Claim 1. All vertices in G have degree 2, except that Type III vertices with $|\sigma| = 1$ or n have degree 1.

Proof. Let σ be a Type I vertex with $\max(\sigma) = |\sigma| + 1 = d$. By Conditions (1) and (4), a neighbor τ of σ must be a vertex of Type II or III and have $\max(\tau) = |\tau| = d$.

Since $|\sigma| + 1 = \max(\sigma)$, there exists a unique index $1 \leqslant j \leqslant d$ such that the elements of σ can be expressed as $A_1 < \cdots < A_{j-1} < A_{j+1} < \cdots < A_d$, where $|A_i| = i$ for all i.

If $1 \leqslant j < d$, then there exist two indices $1 \leqslant t, r \leqslant d$ such that the t-th and the r-th terms are non-zero in A_{j+1} (denoted by a_t and a_r, respectively), but zero in A_{j-1} (or A_{j-1} does not exist in case $j = 1$). Let $\tau_1 = \sigma \cup A_j$ and $\tau_2 = \sigma \cup A'_j$, where A_j (or A'_j, respectively) is obtained by replacing the t-th (or r-th, respectively) term of A_{j+1} by 0. Since σ is agreeable alternating and there are no complementary pairs, each of τ_1 and τ_2 is a Type II or III vertex, and they are the only neighbors of σ in G.

If $j = d$, then $\sigma = A_1 < \cdots < A_{d-1}$, and $|A_i| = i$. Since $\max(\sigma) = d$, there exists a unique index $1 \leqslant t < d$ such that the t-th term of all elements of σ is 0. Hence, the only two neighbors of σ are $\tau : A_1 < \cdots < A_{d-1} < A_d$, where A_d is either $(A^+_{d-1} \cup \{t\}, A^-_{d-1})$ or $(A^+_{d-1}, A^-_{d-1} \cup \{t\})$. Similar to the above discussion, each τ is a Type II or III vertex.

Let σ be a Type II vertex. By (1) and (2), its neighbors τ must be alternating simplexes obtained from σ by deleting one element. Since σ is almost-alternating, there are exactly two elements such that the deletion of each from σ results in an alternating simplex. Since σ is agreeable, each of these two resulted alternating simplexes τ is either a vertex of Type I (if $\max(\tau) = \max(\sigma)$) or a vertex of Type III (if $\max(\tau) = \max(\sigma) - 1$). Both are neighbors of σ.

Let σ be a Type III vertex. By (1), a neighbor τ of σ has $|\tau| = |\sigma| \pm 1$. Of course, if $|\sigma| = 1$, then no neighbor τ of σ has $|\tau| = |\sigma| - 1$; if $|\sigma| = n$, then no neighbor τ of σ has $|\tau| = |\sigma| + 1$. Now we show that if $|\sigma| \geqslant 2$ (respectively, $|\sigma| \leqslant n - 1$) then σ has exactly one neighbor τ with $|\tau| = |\sigma| - 1$ (respectively, with $|\tau| = |\sigma| + 1$).

Assume $|\sigma| \geqslant 2$. If σ is agreeable, then delete the element of σ with the maximum absolute label in $\lambda(\sigma)$. If σ is not agreeable, then delete the element with the minimum absolute label in $\lambda(\sigma)$. For each of the two cases, if the resulted simplex τ has $\max(\tau) = \max(\sigma)$, then τ is agreeable (since σ is agreeable) so it is a vertex of Type I. If τ has $\max(\tau) = \max(\sigma) - 1$, then τ is a vertex of Type III. In both cases, τ is a neighbor of σ. By (2) and (3), the deletion of any other element from σ is not a neighbor of σ.

Now consider $|\sigma| \leqslant n - 1$. Denote $\sigma = A_1 < A_2 < \ldots < A_d$, where $d \leqslant n - 1$ and $A_d = (a_1, \ldots, a_d, 0, \ldots, 0)$. Let $A_{d+1} = (a_1, \ldots, a_d, \operatorname{sign}(\sigma), 0, \ldots, 0)$. Then $\tau = A_1 < \cdots < A_d < A_{d+1}$ is a vertex of Type II or III, and is a neighbor of σ. By (3) and (4), τ is the only neighbor of σ with an additional element.

In conclusion, each Type III vertex has degree 2 if $2 \leqslant d \leqslant n - 1$, and degree 1 if $d = 1, n$. This completes the proof of Claim 1. □

By Claim 1, G is a union of disjoint paths and cycles. The vertices of degree 1 are $\{(1, 0, \ldots, 0)\}$, $\{(-1, 0, \ldots, 0)\}$, and all alternating simplexes of size n. For each path $P = (\sigma_1, \sigma_2, \ldots, \sigma_t)$ in G, its *negation* $-P = (-\sigma_1, -\sigma_2, \ldots, -\sigma_t)$ is also a path in G. Here $-\sigma_i$ is the set obtained from σ_i by negating each of its elements. Observe that $P \neq -P$, for otherwise, we must have $\sigma_t = -\sigma_1$, $\sigma_{t-1} = -\sigma_2$, and eventually we get either $\sigma_i = -\sigma_i$ or $\sigma_{i+1} = -\sigma_i$. Both are impossible. Hence the paths in G come in

pairs, resulting in an even number of paths in G. So G has $4r$ vertices of degree 1, for some $r \geqslant 1$. Thus there are $4r-2$ alternating simplexes of size n. Observe that if σ is a positive alternating simplex, then $-\sigma$ is a negative alternating simplex. Hence there are $2r-1$ positive alternating simplexes of size n. This completes the proof for Lemma 4. □

Note that without Condition (4) in the above proof, Claim 1 does not hold. However, this condition was missing in the proof presented in [19], but was added in [26].

3 Proof of Lemma 2

We prove Lemma 2 by the same labeling used in [3]. However, the argument is further simplified. Let c be a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$ using colors from the set $\{2k-1, 2k, \ldots, n\}$. For a subset A of $[n]$ with $|A| \geqslant k$, let

$$c(A) = \max\{c(U) \colon U \subseteq A, |U| = k\}.$$

Let $\prec$ be an arbitrary linear ordering of $2^{[n]}$ such that if $|X| < |Y|$, then $X \prec Y$. Let λ be a labeling from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm n\}$ defined by:

$$\lambda(A) = \begin{cases} |A|, & \text{if } |A| \leqslant 2k-2 \text{ and } A^- \prec A^+; \\ -|A|, & \text{if } |A| \leqslant 2k-2 \text{ and } A^+ \prec A^-; \\ c(A^+), & \text{if } |A| \geqslant 2k-1 \text{ and } A^- \prec A^+; \\ -c(A^-), & \text{if } |A| \geqslant 2k-1 \text{ and } A^+ \prec A^-. \end{cases}$$

Notice that if $|A| \geqslant 2k-1$, then $|A^+| \geqslant k$ or $|A^-| \geqslant k$. Hence, λ is well-defined. Apparently, λ is anti-podal. Suppose there exists a complementary pair $X < Y$ with $\lambda(X) = -\lambda(Y)$. That is, $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$, where $X^+ \subseteq Y^+$, $X^- \subseteq Y^-$, and it is not the case that $X^+ = Y^+$ and $X^- = Y^-$. As $X < Y$, so $|X| < |Y|$. Assume $\lambda(X) > 0$. (The other case is similar.) By definition of λ, it must be $|X|, |Y| \geqslant 2k-1$. Therefore, there exist $A, B \subseteq [n]$ such that $|A| = |B| = k$, $A \subseteq X^+ \subseteq Y^+$, $B \subseteq Y^-$, and $c(A) = c(B)$, which is impossible as $A \cap B = \emptyset$ (since $Y^+ \cap Y^- = \emptyset$). Thus there are no complementary pairs. By Lemma 4, there are an odd number of positive alternating simplexes of size n.

Claim 2. Assume $\sigma : X_1 < X_2 < \cdots < X_n$ is a positive alternating simplex with respect to λ. Then $|X^+_{2k-2}| = |X^-_{2k-2}| = k-1$, and $[n]$ can be partitioned as $[n] = X^+_{2k-2} \cup X^-_{2k-2} \cup \{a_{2k-1}, a_{2k}, \ldots, a_n\}$, where

$$\begin{array}{ll} c(X^+_{2k-2} \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_j\}) = j, & \text{if } j \text{ is odd;} \\ c(X^-_{2k-2} \cup \{a_{2k}, a_{2k+2}, \ldots, a_j\}) = j, & \text{if } j \text{ is even.} \end{array}$$

Proof. By assumption, $\lambda(\sigma) = \{1, -2, \ldots, (-1)^{n-1}n\}$. So, $|X_i| = i$ for $1 \leqslant i \leqslant n$. By definition of λ, $\lambda(X_i) = (-1)^{i-1}i$ for $1 \leqslant i \leqslant 2k-2$, $|X^+_{2k-2}| = |X^-_{2k-2}| = k-1$, and $\lambda(\{X_{2k-1}, \ldots, X_n\}) = \{2k-1, -2k, \ldots, (-1)^{n-1}n\}$.

Let $q = \lceil \frac{n-2k+2}{2} \rceil$ and $q' = \lfloor \frac{n-2k+2}{2} \rfloor$. The set $\lambda(\{X_{2k-1}, \ldots, X_n\})$ consists of q positive labels and q' negative labels. By the definition of λ, if $\lambda(X_i)$ is positive (respectively, negative), X_i is obtained from X_{i-1} by adding one element to X^+_{i-1} (respectively, to X^-_{i-1}). Thus when i changes from $2k-1$ to n, the sets X^+_i (respectively, X^-_i) changed q times (respectively, q' times), each time a new element is added. Since the positive (respectively, negative) labels in $\lambda(\{X_{2k-1}, \ldots, X_n\})$ are $\{2k-1, 2k+1, \ldots, 2(k+q-1)-1\}$ (respectively, $\{-2k, -(2k+2), \ldots, -(2(k+q'-1))\}$), by the monotonicity of c, each time when a new element is added to X^+_i (or X^-_i, respectively), the value of $c(X^+_i)$ (or $c(X^-_i)$) increases by 2. Therefore $\{2k-1, 2k, \ldots, n\}$ is partitioned into $I = \{j_1 < j_2 < \ldots < j_q\}$ and $I' = \{j'_1 < j'_2 < \ldots < j'_{q'}\}$ such that $\lambda(X_{j_t}) = c(X^+_{j_t}) = 2k-2+2t-1$ and $\lambda(X_{j'_t}) = -c(X^-_{j'_t}) = -(2k-2+2t)$. Moreover $X^+_{j_t}$ is obtained from $X^+_{j_{t-1}}$ by adding one element, and $X^-_{j_t}$ is obtained from $X^-_{j_{t-1}}$ by adding one element. So Claim 2 follows. □

Let Γ be the family of vectors X with $|X^+| = |X^-| = k-1$. By Claim 2, each positive alternating simplex of size n contains exactly one element in Γ. For $W \in \Gamma$, let $\alpha(W, \lambda)$ be the number of positive alternating simplexes of size n with respect to λ, containing W as an element. By Lemma 4, $\Sigma_{X\in\Gamma}\alpha(X, \lambda)$ is odd. Hence there exists $Z \in \Gamma$ such that $\alpha(Z, \lambda)$ is odd. Let $\sigma : X_1 < X_2 < \cdots < X_n$ be a positive alternating simplex with respect to λ, where $Z = X_{2k-2}$. Let $Z = (Z^+, Z^-) = (S, T)$.

Define $\lambda' : \mathcal{F}^n \to \{\pm 1, \pm 2, \ldots, \pm n\}$ by

$$\lambda'(X) = \begin{cases} -\lambda(X), & \text{if } X \in \{Z, -Z\}; \\ \lambda(X), & \text{otherwise.} \end{cases}$$

Similar to λ, λ' is also anti-podal without complementary pairs. Moreover, Claim 2 holds for λ'. By Lemma 4, $\Sigma_{X\in\Gamma}\alpha(X, \lambda')$ is odd. Since $\alpha(X, \lambda') = \alpha(X, \lambda)$ for $X \in \Gamma \setminus \{Z, -Z\}$, so $\alpha(Z, \lambda) + \alpha(-Z, \lambda) \equiv \alpha(Z, \lambda') + \alpha(-Z, \lambda')$ (mod 2). Because $\lambda(-Z) = 2k-2 = \lambda'(Z)$, we get $\alpha(-Z, \lambda) = \alpha(Z, \lambda') = 0$, implying $\alpha(-Z, \lambda') \equiv \alpha(Z, \lambda) \equiv 1$ (mod 2). Hence, there exists a positive alternating simplex $\tau : Y_1 < \cdots < Y_n$ with respect to λ', where $Y_{2k-2} = -Z = (T, S)$. Apply Claim 2 to σ and τ, we obtain for $2k-1 \leqslant i \leqslant n$:

$$\begin{aligned} &c(S \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_i\}) = c(T \cup \{b_{2k-1}, b_{2k+1}, \ldots, b_i\}) = i, && \text{for odd } i; \\ &c(T \cup \{a_{2k}, a_{2k+2}, \ldots, a_i\}) = c(S \cup \{b_{2k}, b_{2k+2}, \ldots, b_i\}) = i, && \text{for even } i, \end{aligned}$$

where $\{a_{2k-1}, a_{2k}, \ldots, a_n\} = \{b_{2k-1}, b_{2k}, \ldots, b_n\} = [n] \setminus (S \cup T)$.

To complete the proof for Lemma 2, it remains to show: For any index $2k-1 \leqslant i \leqslant n$, it holds that $a_i = b_i$ and $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$. We verify this by induction

on i. Assume $i = 2k-1$. As $c(S \cup \{a_{2k-1}\}) = c(T \cup \{b_{2k-1}\}) = 2k-1$, so $S \cup \{a_{2k-1}\}$ and $T \cup \{b_{2k-1}\}$ are not adjacent, implying $a_{2k-1} = b_{2k-1}$. Similarly, it holds for $i = 2k$.

Assume $i \geqslant 2k+1$ and the result holds for $j < i$. If i is odd, as $S \cup \{a_i\}$ is adjacent to $T \cup \{a_j\}$ for all $2k-1 \leqslant j < i$, it follows that $c(S \cup \{a_i\}) \neq c(T \cup \{a_j\}) = j$ for $2k-1 \leqslant j < i$. Thus, $c(S \cup \{a_i\}) = i$, as $c(S \cup \{a_i\}) \leqslant i$. Similarly, we get $c(T \cup \{b_i\}) = i$. Hence, $S \cup \{a_i\}$ and $T \cup \{b_i\}$ are not adjacent, implying $a_i = b_i$. The case for even i is obtained similarly. This completes the proof for Lemma 2. □

Note that according to (1.1), Theorem 3 implies the Lovász-Kneser Theorem. Moreover, Lovász-Kneser Theorem can be derived directly from Lemma 4. Assume to the contrary, $\chi(\mathrm{KG}(n,k)) \leqslant n-2k+1$. Let c be a proper coloring for $\mathrm{KG}(n,k)$ using colors from $\{2k-1, 2k, \ldots, n-1\}$. Let λ be the same labeling defined in our proof, except in this case λ is from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm(n-1)\}$, instead of to $\{\pm 1, \pm 2, \ldots, \pm n\}$. By the same argument, λ is anti-podal without complementary pairs, contradicting Lemma 4 (as $n-1 < n$).

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References

[1] N. Alon, P. Frankl, L. L. Lovász. The chromatic number of Kneser hypergraphs. *Trans. Amer. Math. Soc.*, 298:359–370, 1986.

[2] I Bárány. A short of Kneser's conjecture. *J. Combin. Theory Ser. A*, 25:325–326, 1978.

[3] G. J. Chang, D. D.-F. Liu, and X. Zhu. A short proof for Chen's Alternative Kneser Coloring Lemma. *J. Combin. Theory Ser. A*, 120:159–163, 2013.

[4] P.-A. Chen. A new coloring theorem of Kneser graphs. *J. Combin. Theory Ser. A*, 118(3):1062–1071, 2011.

[5] K. Fan. A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math. (2)*, 56:431–437, 1952.

[6] R. M. Freund and M. J. Todd. A constructive proof of Tucker's combinatorial lemma. *J. Combin. Theory Ser. A*, 30:321–325, 1981.

[7] J. Greene. A new short proof of Kneser's conjecture. *Amer. Math. Monthly*, 109:918-920, 2002.

[8] H. Hajiabolhassan and A. Taherkhani. Graph powers and graph homomorphisms. *Electron. J. Combin.* 17, no. 1, Research Paper 17, 16 pp., 2010.

[9] H. Hajiabolhassan and X. Zhu. Circular chromatic number of Kneser graphs. *J. Combin. Theory Ser. B*, 88(2):299–303, 2003.

[10] A. Johnson, F. C. Holroyd, and S. Stahl. Multichromatic numbers, star chromatic numbers and Kneser graphs. *J. Graph Theory*, 26(3):137–145, 1997.

[11] M. Kneser. Aufgabe 300. *Jber. Deutsch. Math.-Verein.*, 58:27, 1955.

[12] I. Kriz. Equivalent cohomology and lower bounds for chromatic numbers. *Trans. Amer. Math. Soc.*, 333:567–577, 1992.

[13] I. Kriz. A corretion to "Equivalent cohomology and lower bounds for chromatic numbers". *Trans. Amer. Math. Soc.*, 352:1951–1952, 2000.

[14] K.-W. Lih and D. D.-F. Liu. Circular chromatic numbers of some reduced Kneser graphs. *J. Graph Theory*, 41: 62–68, 2002.

[15] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[16] J. Matoušek. Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry. *Springer*, 2003.

[17] J. Matoušek. A combinatorial proof of Kneser's conjecture. *Combinatorica*, 24:163–170, 2004.

[18] F. Meunier. A topological lower bound for the circular chromatic number of Schrijver graphs. *J. Graph Theory*, 49(4):257–261, 2005.

[19] T. Prescott and F. Su. A constructive proof of Ky Fan's generalization of Tucker's lemma. *J. Combin. Theory Ser. A*, 111:257–265, 2005.

[20] K. S. Sarkaria. A generalized Kneser conjecture. *J. Combin. Theory Ser. B*, 49:236–240, 1990.

[21] A. Schrijver. Vertex-critical subgraphs of Kneser graphs. *Nieuw Arch. Wiskd., III. Ser*, 26:454–461, 1978.

[22] G. Simonyi and G. Tardos. Local chromatic number, Ky Fan's theorem and circular colorings. *Combinatorica*, 26(5):587–626, 2006.

[23] A. W. Tucker. Some topological properties of disk and sphere. *Proc. First Canadian Math. Congr., Montreal*, Toronto Press, 285–309, 1946.

[24] X. Zhu. Circular chromatic number: a survey. *Discrete Math.*, 229(1-3):371–410, 2001.

[25] X. Zhu. Recent developments in circular colouring of graphs. *Topics in discrete mathematics*, 26: 497–550, Algorithms Combin., Springer, Berlin, 2006.

[26] X. Zhu. Circular coloring and flow. Lecture note, 2012.

[27] G. Ziegler. Generalized Kneser coloring theorems with combinatorial proofs. *Invent Math.*, 147:671–691, 2002.