# A Krull-Schmidt Theorem for Noetherian Modules*

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We prove a version of the Krull-Schmidt Theorem which applies to Noetherian modules. As a corollary we get the following cancellation rule: If $A, B, C$ are nonzero Noetherian modules such that either $A \oplus C \cong B \oplus C$, or $A^n \cong B^n$ for some $n \in \mathbb{N}$, then there are modules $A' \leq A$ and $B' \leq B$ such that $A' \cong B'$ and $\operatorname{len} A' = \operatorname{len} A = \operatorname{len} B' = \operatorname{len} B$. Here the ordinal valued length, $\operatorname{len} A$, of a module $A$ is as defined in [3] and [5]. In particular, $A$, $B$, $A'$ and $B'$ have the same Krull dimension, and $A/A'$ and $B/B'$ have strictly smaller Krull dimension than $A$ and $B$.

## 1. INTRODUCTION

An old and important problem of module theory is the direct sum cancellation question: Suppose that $A, B, C$ are left modules over a ring $R$ such that $A \oplus C \cong B \oplus C$. What can be said about the relationship between $A$ and $B$? In particular, are $A$ and $B$ isomorphic? If this happens we say that direct sum cancellation has occurred.

In general $A$ and $B$ can be quite different. For example, $A$ and $C$ could be infinite dimensional vector spaces and $B = 0$. If, however, we require $C$ satisfy a chain condition then we get the following two contrasting results:

THEOREM 1.1. *Let $A, B, C$ be modules such that $A \oplus C \cong B \oplus C$.*

*1. [4] If $C$ is Artinian, then $A \cong B$.*

*2. [2] If $C$ is Noetherian, then $A$ and $B$ have isomorphic submodule series. That is, there are series $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ and $0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$ and a permutation $\sigma$ of the indices such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for $i = 1, 2, \ldots, n$.*

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This suggests that the cancellation question is complicated for Noetherian modules, and indeed there are easy examples ([2], [7]) of Noetherian modules $A, B, C$ such that $A \oplus C \cong B \oplus C$, but $A \ncong B$.

On the positive side there are certain narrow circumstances in which direct sum cancellation does occur for Noetherian modules. An example of this is cancellation within a genus class: Suppose $R$ is a commutative Noetherian reduced ring of dimension 1, and $A, B, C$ are Noetherian modules all in the same genus class of $R$, then $A \oplus C \cong B \oplus C$ implies $A \cong B$. For this result and its generalizations see [6].

In this paper we get new information about the direct sum cancellation question for Noetherian modules by showing how the Krull-Schmidt Theorem applies to such modules.

The main tool is an ordinal valued measure of the size of Noetherian modules which generalizes both Krull dimension and length. Specifically, for a Noetherian module $A$, its ordinal valued length, $\operatorname{len} A$, has the usual meaning if $A$ has finite length, and if $\operatorname{len} A$ is written in normal form, $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are ordinals, then the Krull dimension of $A$ is $\gamma_1$.

In terms of this ordinal valued length, our main cancellation result is the following:

THEOREM 1.2. *Let $A, B$ and $C$ be Noetherian left modules over a ring $R$ such that either $A \oplus C \cong B \oplus C$, or $A^n \cong B^n$ for some $n \in \mathbb{N}$. Then there are essential submodules $A' \leq A$ and $B' \leq B$ such that $A' \cong B'$ and $\operatorname{len} A' = \operatorname{len} A = \operatorname{len} B' = \operatorname{len} B$. In particular, $A$, $B$, $A'$ and $B'$ have the same Krull dimension, and, if $A$ and $B$ are nonzero, $A/A'$ and $B/B'$ have strictly smaller Krull dimension than $A$ and $B$.*

This theorem is an interesting contrast to Theorem 1.1(2): Given Noetherian modules $A, B$ and $C$ such that $A \oplus C \cong B \oplus C$, Theorem 1.1(2) guarantees the existence of isomorphic submodule series in $A$ and $B$ but provides no indication of the number of factors, the permutation $\sigma$ or the size of the subfactor modules. Theorem 1.2, on the other hand, provides a matchup of submodules of $A$ and $B$ of a specific size. Perhaps some combination of the techniques used to prove Theorems 1.2 and 1.1(2) can be used to provide an even more precise description of the relationship between the modules $A$ and $B$.

## 2. MAIN RESULTS

As already explained, the cancellation results we prove come from the application of the Krull-Schmidt Theorem to the category of Noetherian modules. We will use the following category theoretic formulation of this

theorem (more properly called the Krull-Schmidt-Remak-Azumaya Theorem):

THEOREM 2.1. *[1, Ch. 1, 3.6] Let $\mathcal{C}$ be an additive category in which idempotents split, and $A$ an object of $\mathcal{C}$.*

*1. If $\mathrm{End}_{\mathcal{C}} A$ is local then $A$ is indecomposable in $\mathcal{C}$.*

*2. If $A$ is a direct sum of objects with local endomorphism rings, then any two direct sum decompositions of $A$ into indecomposable objects are isomorphic.*

Here we mean that two direct sum decompositions

$$A \cong A_1 \oplus A_2 \oplus \ldots \oplus A_n \cong B_1 \oplus B_2 \oplus \ldots \oplus B_m$$

are **isomorphic** if $n = m$ and there is a permutation $\sigma$ of the indices such that $A_i \cong B_{\sigma(i)}$ for $i = 1, 2, \ldots, n$.

For example, suppose $\mathcal{C} = R$-**Noeth**, the category of Noetherian left modules over a ring $R$. Any module in $\mathcal{C}$ can be written as a direct sum of indecomposable modules. Moreover, using Fitting's Lemma, one can show that any finite length indecomposable module has a local endomorphism ring [10, 2.9.8]. Thus we get the usual Krull-Schmidt Theorem: Any finite length module has a unique (up to isomorphism of decompositions) direct sum decomposition into indecomposable modules. As an immediate consequence we get direct sum cancellation for finite length modules: If $A, B, C$ are finite length modules such that $A \oplus C \cong B \oplus C$ then $A \cong B$. We also get a multiplicative cancellation rule: If $A, B$ are finite length modules such $A^n \cong B^n$ for some $n \in \mathbb{N}$, then $A \cong B$.

The Krull-Schmidt Theorem does not apply to the entire category of Noetherian modules because the endomorphism ring of a Noetherian indecomposable module is not necessarily local. To circumvent this difficulty we apply Theorem 2.1 to a new category, $R$-**BNoeth** (see Definition 2.5), whose objects are Noetherian left $R$-modules, but whose morphisms have been changed in such a way that indecomposable objects in the category do, in fact, have local endomorphism rings.

Just as for finite length modules, proving that indecomposable objects in $R$-**BNoeth** have local endomorphism rings proceeds via Fitting's Lemma. In the finite length case, the length of the module in question plays a key role in this lemma. To extend this lemma to Noetherian modules we use the ordinal valued length which is defined as follows [3], [5]: For a Noetherian module $A$, let $\mathcal{L}(A)$ be the lattice of all submodules of $A$. By induction, define a map $\lambda$ from $\mathcal{L}(A)$ to ordinal numbers such that

$$\lambda(A') = \sup\{\lambda(A'') + 1 \mid A' < A''\}$$

for all $A' \in \mathcal{L}(A)$. Note that $\lambda(A) = 0$ and $\lambda$ is a (strictly) decreasing function. Then the **length** of $A$ is $\operatorname{len} A = \lambda(0)$. The module $A$ has finite length if and only if $\operatorname{len} A$ is a finite ordinal, and in this case $\operatorname{len} A$ has the usual meaning.

One important property of ordinal numbers is that any nonzero ordinal $\alpha$ can be expressed uniquely in **Cantor normal form**

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$$

where $\omega$ is the first infinite ordinal, $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ are ordinals and $n_1, n_2, \ldots, n_n \in \mathbb{N}$.

By adding these normal forms as if they were polynomials we get the natural sum operation on ordinals [11, Ch. XIV 28]. More precisely, with suitable re-labeling, the normal forms for nonzero ordinals $\alpha$ and $\beta$ can be written using the same strictly decreasing set of ordinal exponents $\gamma_1 > \gamma_2 > \cdots > \gamma_n$:

$$\alpha = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n \qquad \beta = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$$

where $n_i, m_i \in \{0, 1, 2, 3, \ldots\}$. Then the **natural sum** of $\alpha$ and $\beta$ is

$$\alpha \oplus \beta = \omega^{\gamma_1}(m_1 + n_1) + \omega^{\gamma_2}(m_2 + n_2) + \cdots + \omega^{\gamma_n}(m_n + n_n).$$

In addition, we define $0 \oplus \alpha = \alpha \oplus 0 = \alpha$. The natural sum is associative, commutative and cancellative: $(\alpha \oplus \gamma = \beta \oplus \gamma \implies \alpha = \beta)$ and $(\alpha \oplus \gamma \leq \beta \oplus \gamma \implies \alpha \leq \beta)$; whereas ordinary ordinal addition is associative, not commutative and cancellative only on the left: $(\gamma + \alpha = \gamma + \beta \implies \alpha = \beta)$ and $(\gamma + \alpha \leq \gamma + \beta \implies \alpha \leq \beta)$. For further details about ordinal arithmetic see [9] or [11].

With these facts about ordinal numbers at hand, we can now present the main properties of the ordinal length function of Noetherian modules from [3] and [5]. We write $\operatorname{Kdim} A$ for the Krull dimension of a module $A \in R$-**Noeth**.

THEOREM 2.2. *[5, 2.1, 2.11, 2.3(ii)] Let $A, B, C \in R$-**Noeth**.*

1. *If $0 \to A \to B \to C \to 0$ is an exact sequence, then*

$$\operatorname{len} C + \operatorname{len} A \leq \operatorname{len} B \leq \operatorname{len} C \oplus \operatorname{len} A.$$

2. $\operatorname{len}(A \oplus B) = \operatorname{len} A \oplus \operatorname{len} B$.
3. *If $\operatorname{len} A = \omega^{\gamma_1} n_1 + \cdots + \omega^{\gamma_n} n_n$ in normal form, then $\operatorname{Kdim} A = \gamma_1$.*

From 1 of this theorem we see that the length of a Noetherian module is greater or equal to the length of any of its submodules or factor modules.

From 2 and the cancellative property of $\oplus$, we have that, if $A \oplus C \cong B \oplus C$ or $A^n \cong B^n$ for some $A, B, C \in R\text{-}\mathbf{Noeth}$ and $n \in \mathbb{N}$, then $\operatorname{len} A = \operatorname{len} B$. In particular, from 3, $\operatorname{Kdim} A = \operatorname{Kdim} B$. This result we will strengthen considerably in Theorem 2.9.

The property of finite length modules at the heart of the proof of Fitting's Lemma is that, if $A' \leq A$ and $\operatorname{len} A' = \operatorname{len} A$, then $A' = A$. In general, this is not true of Noetherian modules. Indeed, many Noetherian modules contain proper submodules which are isomorphic to themselves. In this circumstance, the length of the proper submodule is, of course, the same as the length of the whole module. This motivates the following definition:

DEFINITION 2.3. *Given a module $A \in R\text{-}\mathbf{Noeth}$, any submodule $A' \leq A$ such that $\operatorname{len} A' = \operatorname{len} A$ is said to be* **big** *in $A$. This situation is denoted $A' \trianglelefteq A$.*

For an easy example, let $I$ be a nonzero ideal in a Noetherian domain $R$. Then for a nonzero element $x \in I$ we have $R \cong Rx \leq I \leq R$, and so $\operatorname{len} R = \operatorname{len} Rx \leq \operatorname{len} I \leq \operatorname{len} R$. Thus $\operatorname{len} I = \operatorname{len} R$ and $I \trianglelefteq R$.

As we have already noted, if $A$ is a finite length module and $A' \trianglelefteq A$, then $A' = A$. Other important properties of the relation $\trianglelefteq$ are collected in the next lemma.

LEMMA 2.4. *Let $A, A', A'', B, B' \in R\text{-}\mathbf{Noeth}$.*

1. *If $A'' \leq A' \leq A$, then $A'' \trianglelefteq A$ if and only if $A'' \trianglelefteq A'$ and $A' \trianglelefteq A$.*
2. *$\psi\colon A \to B$ and $B' \trianglelefteq B \implies \psi^{-1}(B') \trianglelefteq A$*
3. *$A', A'' \trianglelefteq A \implies A' \cap A'' \trianglelefteq A$*
4. *$A' \trianglelefteq A$ and $B \leq A \implies B \cap A' \trianglelefteq B$*
5. *$A' \trianglelefteq A \implies A'$ is essential in $A$*
6. *$A' \trianglelefteq A \implies \operatorname{Kdim} A' = \operatorname{Kdim} A$.*
7. *$A' \trianglelefteq A$ and $A \neq 0 \implies \operatorname{Kdim}(A/A') < \operatorname{Kdim} A$.*

*Proof.*

1. Immediate from the definition.

2. We have $B' \leq B' + \psi(A) \leq B$, and so $\operatorname{len} B' = \operatorname{len}(\psi(A) + B') = \operatorname{len} B$. Now consider the exact sequence

$$0 \to \psi^{-1}(B') \xrightarrow{\sigma} A \oplus B' \xrightarrow{\tau} \psi(A) + B' \to 0$$

where $\sigma(a) = (a, \psi(a))$ and $\tau(a, b) = \psi(a) - b$ for all $a \in A$ and $b \in B'$. Using Theorem 2.2(1,2), we get

$$\begin{aligned}\operatorname{len} A \oplus \operatorname{len} B' &= \operatorname{len}(A \oplus B') \\ &\leq \operatorname{len}(\psi^{-1}(B')) \oplus \operatorname{len}(\psi(A) + B') = \operatorname{len}(\psi^{-1}(B')) \oplus \operatorname{len} B',\end{aligned}$$

and then cancellation from this inequality yields $\operatorname{len} A \leq \operatorname{len}(\psi^{-1}(B'))$. Since $\psi^{-1}(B') \leq A$, the opposite inequality is, of course, true and we have $\operatorname{len} A = \operatorname{len}(\psi^{-1}(B'))$.

3. Apply 2 to the inclusion map $\psi\colon A' \to A$.

4. Apply 2 to the inclusion map $\psi\colon B \to A$.

5. If $A' \oplus B \leq A$, then $\operatorname{len} A' \oplus \operatorname{len} B = \operatorname{len}(A' \oplus B) \leq \operatorname{len} A$. Since $\operatorname{len} A' = \operatorname{len} A$, we can cancel from this inequality to get $\operatorname{len} B = 0$, that is, $B = 0$.

6. Directly from Theorem 2.2(3).

7. From Theorem 2.2(1) and $\operatorname{len} A' = \operatorname{len} A$, it follows that $\operatorname{len}(A/A') + \operatorname{len} A \leq \operatorname{len} A$. A simple application of ordinal arithmetic to the normal forms for $\operatorname{len}(A/A')$ and $\operatorname{len} A$, together with Theorem 2.2(3) shows that if $\operatorname{Kdim}(A/A') \geq \operatorname{Kdim} A$, then $\operatorname{len}(A/A') + \operatorname{len} A > \operatorname{len} A$ contrary to the above inequality. Thus we must have $\operatorname{Kdim}(A/A') < \operatorname{Kdim} A$. ∎

All the claims implied in the following definition are easy consequences of Lemma 2.4(1-4).

Definition 2.5. *Let $A, B \in R$-**Noeth**. A* **big homomorphism** *from $A$ to $B$ is a pair $(\psi, A')$ where $A' \trianglelefteq A$ and $\psi \in \operatorname{Hom}(A', B)$. Two such big homomorphisms $(\psi_1, A_1)$ and $(\psi_2, A_2)$ are equivalent if there is some $A' \trianglelefteq A_1 \cap A_2$ such that $\psi_1 \mid_{A'} = \psi_2 \mid_{A'}$. The equivalence class containing $(\psi, A')$ will be written $[\psi, A']$, and the set of equivalence classes of big homomorphisms from $A$ to $B$ will be denoted* $\operatorname{BHom}(A, B)$.

*If $[\psi_1, A_1], [\psi_2, A_2] \in \operatorname{BHom}(A, B)$ we define*

$$[\psi_1, A_1] + [\psi_2, A_2] = [\psi_1 + \psi_2, A_1 \cap A_2] \in \operatorname{BHom}(A, B).$$

*If $[\psi, A'] \in \operatorname{BHom}(A, B)$ and $[\phi, B'] \in \operatorname{BHom}(B, C)$, we define*

$$[\phi, B'][\psi, A'] = [\phi \circ \psi, \psi^{-1}(B')] \in \operatorname{BHom}(A, C).$$

*We define the category $R$-**BNoeth** as follows: The objects of $R$-**BNoeth** are the objects of $R$-**Noeth**. If $A, B \in R$-**BNoeth**, the corresponding morphisms are* $\operatorname{BHom}(A, B)$. *Composition and addition of morphisms are as above. The identity morphism in* $\operatorname{BHom}(A, A)$ *is* $[1_A, A]$. *We write* $\operatorname{BEnd} A = \operatorname{BHom}(A, A)$ *which is a ring. To avoid confusion between*

*$R$-**Noeth** and $R$-**BNoeth** we will say that modules $A, B \in R$-**BNoeth** are* **B-isomorphic** *if they are isomorphic in the category $R$-**BNoeth**, and write $A \underset{\mathbf{B}}{\cong} B$. Similarly, $A \in R$-**BNoeth** is* **B-indecomposable** *if it is indecomposable in the category $R$-**BNoeth**.*

We next prove a version of Fitting's Lemma appropriate to the category $R$-**BNoeth**.

LEMMA 2.6 (Fitting). *Let $[\psi, A_1] \in \operatorname{BEnd} A$. For $n = 2, 3, 4, \ldots$, define inductively $A_{n+1} = \psi^{-1}(A_n)$ so that $\psi^n \colon A_n \to A$ and $[\psi^n, A_n] = [\psi, A_1]^n$. Then there is some $n \in \mathbb{N}$ such that*

$$\operatorname{im} \psi^n \oplus \ker \psi^n \trianglelefteq A.$$

*Proof.* Note that if $a \in \ker \psi^n$ then $\psi^n(a) = 0 \in A_1$, so $a \in A_{n+1}$ and $a \in \ker \psi^{n+1}$. Thus we have $\ker \psi \leq \ker \psi^2 \leq \ker \psi^3 \leq \ldots$. Since $A$ is Noetherian there is some $n \in \mathbb{N}$ such that $\ker \psi^n = \ker \psi^m$ for all $m \geq n$. In particular, $\ker \psi^n = \ker \psi^{2n}$. It follows easily that $\operatorname{im} \psi^n \cap \ker \psi^n = 0$, so $\operatorname{im} \psi^n \oplus \ker \psi^n \leq A$ and, of course, $\operatorname{len}(\operatorname{im} \psi^n \oplus \ker \psi^n) \leq \operatorname{len} A$. Applying Theorem 2.2 to the exact sequence $0 \to \ker \psi^n \to A_n \to \operatorname{im} \psi^n \to 0$, we get

$$\operatorname{len} A = \operatorname{len} A_n \leq \operatorname{len}(\ker \psi^n) \oplus \operatorname{len}(\operatorname{im} \psi^n) = \operatorname{len}(\operatorname{im} \psi \oplus \ker \psi).$$

Thus $\operatorname{len}(\operatorname{im} \psi^n \oplus \ker \psi^n) = \operatorname{len} A$. ∎

THEOREM 2.7. *For a ring $R$, $R$-**BNoeth** is an additive category such that:*

*1. A morphism $[\psi, A'] \in \operatorname{BHom}(A, B)$ is monic if and only if $\psi$ is injective.*

*2. A morphism $[\psi, A'] \in \operatorname{BHom}(A, B)$ is epic if and only if $\psi(A'') \trianglelefteq B$ for all $A'' \trianglelefteq A'$.*

*3. A morphism $[\psi, A'] \in \operatorname{BHom}(A, B)$ is an isomorphism if and only if it is both monic and epic, if and only if $\psi$ is injective and $\operatorname{len} A = \operatorname{len} B$.*

*4. If $A, B \in R$-**BNoeth**, then $A \underset{\mathbf{B}}{\cong} B$ if and only if there are $A' \trianglelefteq A$ and $B' \trianglelefteq B$ such that $A' \cong B'$. In particular, a module is **B**-isomorphic with any of its big submodules.*

*5. Let $A_1, A_2, \ldots, A_n \in R$-**BNoeth** and $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$. For $i = 1, 2, \ldots, n$, let $\pi_i \colon A \to A_i$ and $\iota_i \colon A_i \to A$ be the usual projection and injection homomorphisms. Then the module $A$ together with the morphisms $[\pi_i, A] \in \operatorname{BHom}(A, A_i)$ is a product of $A_1, A_2, \ldots, A_n$ in $R$-**BNoeth**, and $A$ together with the morphisms $[\iota_i, A_i] \in \operatorname{BHom}(A_i, A)$ is a coproduct of $A_1, A_2, \ldots, A_n$ in $R$-**BNoeth**.*

*6. Idempotents split in* $R$-**BNoeth**.

*7.* $A \in R$-**BNoeth** *is* **B***-indecomposable if and only if* $A_1 \oplus A_2 \trianglelefteq A$ *implies either* $A_1 = 0$ *or* $A_2 = 0$*, if and only if* $\operatorname{BEnd} A$ *is local.*

*8. Any* $0 \neq A \in R$-**BNoeth** *has a decomposition* $A_1 \oplus A_2 \oplus \ldots \oplus A_n \trianglelefteq A$ *such that each* $A_i$ *is* **B***-indecomposable. In particular,* $A$ *is* **B***-isomorphic to a finite direct sum of* $B$*-indecomposable modules.*

For the convenience of the reader we provide abbreviated definitions of the relevant category theoretic notions using the labels used in the following proof: A category $\mathcal{C}$ is **preadditive** if for all objects $A, B \in \mathcal{C}$, $\operatorname{Hom}(A, B)$ is an Abelian group whose operation distributes over composition of morphisms. A category $\mathcal{C}$ is **additive** if it is preadditive and any pair of objects of $\mathcal{C}$ has a direct sum. A morphism $f \in \operatorname{Hom}(A, B)$ is **monic** (**epic**) if $fg = fh$ ($gf = hf$) implies $g = h$, for all morphisms $g, h$. A morphism $f \in \operatorname{Hom}(A, B)$ is an **isomorphism** if $f$ has a two-sided inverse, meaning there is some $g \in \operatorname{Hom}(B, A)$ such that $fg = 1_B$ and $gf = 1_A$. A morphism $e \in \operatorname{End} A$ is an **idempotent** if $e^2 = e$, and then $e$ **splits** if there are $p \in \operatorname{Hom}(A, B)$ and $q \in \operatorname{Hom}(B, A)$ for some $B \in \mathcal{C}$ such that $qp = e$ and $pq = 1_B$. An object $A \in \mathcal{C}$ is **indecomposable** if $A \cong A_1 \oplus A_2$ implies $A_1 = 0$ or $A_2 = 0$.

*Proof.* For $A, B \in R$-**BNoeth**, it is clear that $\operatorname{BHom}(A, B)$ is an Abelian group under addition which satisfies the distributive laws with respect to compositions. Thus $R$-**BNoeth** is preadditive. In 5 we will show that $R$-**BNoeth** has finite direct sums (that is, coproducts), and so $R$-**BNoeth** is additive.

1. Suppose $f = [\psi, A'] \in \operatorname{BHom}(A, B)$ is monic. Let $C = \ker \psi$ and $g, h \in \operatorname{BHom}(C, A)$ be $g = [1_C, C]$ and $h = [0, C]$. Then $fg = fh = [0, C]$ and so $g = h$. This means that there is some $C' \trianglelefteq C$ such that the identity map and zero map coincide on $C'$. This implies $C' = 0$ and since $\operatorname{len} C = \operatorname{len} C'$, we have $C = 0$ and $\psi$ is injective.

Suppose $\psi$ is injective and we have $g = [\gamma_1, C_1]$ and $h = [\gamma_2, C_2]$ in $\operatorname{BHom}(C, A)$ such that $fg = fh$. Then there is some $C' \trianglelefteq C$ on which $\psi \circ \gamma_1 = \psi \circ \gamma_2$ coincide. Since $\psi$ is injective, we have $\gamma_1 = \gamma_2$ on $C'$ and so $g = h$.

2. Suppose $f = [\psi, A'] \in \operatorname{BHom}(A, B)$ is epic. Let $C = B/\psi(A')$ and $\sigma\colon B \to C$ be the quotient homomorphism. Let $g = [\sigma, B]$ and $h = [0, B]$ which are morphisms in $\operatorname{BHom}(B, C)$. Then $gf = hf = [0, A']$ and so $g = h$. This means that there is some $B' \trianglelefteq B$ such that $\sigma(B') = 0$, that is, $B' \leq \psi(A')$. Thus, in fact, $B' \trianglelefteq \psi(A') \trianglelefteq B$.

If we have $A'' \trianglelefteq A'$, then $f = [\psi, A'] = [\psi, A'']$ and so the same argument shows $\psi(A'') \trianglelefteq B$.

Suppose that $\psi(A'') \trianglelefteq B$ for all $A'' \trianglelefteq A'$ and we have $g = [\gamma_1, B_1]$ and $h = [\gamma_2, B_2]$ in $\mathrm{BHom}(B, C)$ such that $gf = hf$. Then there is some $A'' \trianglelefteq \psi^{-1}(B_1) \cap \psi^{-1}(B_2) \trianglelefteq A'$ on which $\gamma_1 \circ \psi = \gamma_1 \circ \psi$. Hence $\gamma_1$ and $\gamma_2$ coincide on $\psi(A'') \trianglelefteq B$, and so $g = h$.

3. If $\psi$ is injective and $\operatorname{len} A = \operatorname{len} B$, then $\operatorname{len} \psi(A') = \operatorname{len} B$ and so $\psi(A') \trianglelefteq B$. It is then easy to see that $g = [\psi^{-1}, \psi(A')] \in \mathrm{BHom}(B, A)$ is a two-sided inverse of $[\psi, A']$. The remaining claims are immediate from 1 and 2.

4. If $[\psi, A'] \in \mathrm{BHom}(A, B)$ is an isomorphism, then from 3, we have $A' \cong \psi(A') \trianglelefteq B$. Conversely, if $A' \trianglelefteq A$, $B' \trianglelefteq B$ and $\psi\colon A' \to B'$ is an isomorphism, then $[\psi, A'] \in \mathrm{BHom}(A, B)$ is an isomorphism.

5. This can be checked directly, or by [8, page 18, Theorem 1.2], it suffices to notice that

$$\sum_{i=1}^{n} [\iota_i, A_i][\pi_i, A] = [1_A, A] \quad \text{and} \quad [\pi_i, A][\iota_j, A_j] = \begin{cases} [0, A_j] & \text{if } i \neq j \\ [1_{A_i}, A_i] & \text{if } i = j \end{cases}.$$

6. Let $e^2 = e = [\epsilon, A'] \in \mathrm{BEnd}\, A$ be an idempotent. Then $\epsilon = \epsilon^2$ on some $A'' \trianglelefteq A$. Set $p = [\epsilon, A''] \in \mathrm{BHom}(A, B)$ and $q = [\iota, B] \in \mathrm{BHom}(B, A)$ where $B = \epsilon(A'')$ and $\iota\colon B \to A$ is the inclusion map. Then $\iota \circ \epsilon = \epsilon$ on $A''$, which implies $qp = e$, and $\epsilon \circ \iota$ is the identity on $B \cap \epsilon^{-1}(A'') \trianglelefteq B$ and so $pq = [1_B, B]$.

7. If $A$ is **B**-indecomposable and $A_1 \oplus A_2 \trianglelefteq A$, then $A_1 \oplus A_2 \underset{\mathbf{B}}{\cong} A$ and so either $A_1 = 0$ or $A_2 = 0$.

Next suppose that $A_1 \oplus A_2 \trianglelefteq A$ implies $A_1 = 0$ or $A_2 = 0$, and we have $[\psi, A'] \in \mathrm{BEnd}\, A$. From Lemma 2.6, there is some $n \in \mathbb{N}$ such that $\operatorname{im} \psi^n \oplus \ker \psi^n \trianglelefteq A$. Thus either $\operatorname{im} \psi^n = 0$ and $[\psi, A']$ is nilpotent, or $\ker \psi^n = 0$ and $[\psi, A']$ is invertible by 3. Since every element of $\mathrm{BEnd}\, A$ is invertible or nilpotent, $\mathrm{BEnd}\, A$ is a local ring [10, 2.9.9]. The remaining implication is from Theorem 2.1(1).

8. The proof that $A$ has such a decomposition into **B**-indecomposables follows the usual pattern: If $A$ is **B**-indecomposable, then we are done. Otherwise, there are nonzero $A_{11}, A_{12} \leq A$ such that $A_{11} \oplus A_{12} \trianglelefteq A$. If both $A_{11}$ and $A_{12}$ are **B**-indecomposable, we are done. Otherwise we can decompose one of $A_{11}$ or $A_{12}$ and then there are nonzero submodules $A_{21}, A_{22}, A_{23} \leq A$ such that $A_{21} \oplus A_{22} \oplus A_{23} \trianglelefteq A$. Failure of this process to stop would give an infinite direct sum in $A$. ∎

We now have everything we need to apply Theorem 2.1 to the category $R$-**BNoeth**: Any indecomposable object (i.e., **B**-indecomposable module) has a local endomorphism ring, and, any nonzero object has a direct sum decomposition into indecomposables. It follows from Theorem 2.1 that

such decompositions are unique (up to isomorphism of decompositions in $R$-**BNoeth**). Interpreting this in $R$-**Noeth** we get the following:

THEOREM 2.8. *Let $0 \neq A \in R$-**Noeth**. Then there is a decomposition $A_1 \oplus A_2 \oplus \ldots \oplus A_n \trianglelefteq A$ such that each $A_i$ is $\mathbf{B}$-indecomposable, and if $B_1 \oplus B_2 \oplus \ldots \oplus B_m \trianglelefteq A$ is another such decomposition, then $n = m$ and there is a permutation $\sigma$ of the indices such that $A_i \underset{\mathbf{B}}{\cong} B_{\sigma(i)}$ for $i = 1, 2, \ldots, n$.*

Just as for finite length modules, we get cancellation rules in $R$-**BNoeth**: If $A, B, C \in R$-**BNoeth** are such that either $A \oplus C \underset{\mathbf{B}}{\cong} B \oplus C$, or $A^n \underset{\mathbf{B}}{\cong} B^n$ for some $n \in \mathbb{N}$, then $A \underset{\mathbf{B}}{\cong} B$. Interpreting this in $R$-**Noeth** we get, in a weakened, but perhaps more useful form, the following:

THEOREM 2.9. *If $A, B, C \in R$-**Noeth** are such that either $A \oplus C \cong B \oplus C$, or $A^n \cong B^n$ for some $n \in \mathbb{N}$, then there are big submodules $A' \trianglelefteq A$ and $B' \trianglelefteq B$ such that $A' \cong B'$.*

Theorem 1.2 in the introduction is just a restatement of this theorem noting that $A' \trianglelefteq A$ and $B' \trianglelefteq B$ with $A' \cong B'$ implies that $\operatorname{len} A = \operatorname{len} A' = \operatorname{len} B' = \operatorname{len} B$. From Lemma 2.4(6,7) we have that $\operatorname{Kdim} A = \operatorname{Kdim} A' = \operatorname{Kdim} B' = \operatorname{Kdim} B$, and if $A, B \neq 0$, then $\operatorname{Kdim}(A/A') < \operatorname{Kdim} A$ and $\operatorname{Kdim}(B/B') < \operatorname{Kdim} B$. From Lemma 2.4(5) we have also that $A'$ is essential in $A$ and $B'$ is essential in $B$.

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