# A Short Proof for Chen's Alternative Kneser Coloring Lemma

Gerard Jennhwa Chang* Daphne Der-Fen Liu† Xuding Zhu‡

July 9, 2012

**Abstract**

We give a short proof for Chen's Alternative Kneser Coloring Lemma. This leads to a short proof for the Johnson-Holroyd-Stahl conjecture that Kneser graphs have their circular chromatic numbers equal to their chromatic numbers.

## 1 Introduction

Suppose $G$ is a graph and $p \geq q \geq 1$ are integers. A $(p,q)$-*coloring* of $G$ is a mapping $c\colon V(G) \to \{0,1,\ldots,p-1\}$ such that $q \leq |f(x)-f(y)| \leq p-q$ for every edge $xy$ of $G$. A graph is $(p,q)$-*colorable* if it admits a $(p,q)$-coloring. The *circular chromatic number* of $G$ is

$$\chi_c(G) = \inf\{p/q\colon\ G \text{ is } (p,q)\text{-colorable}\}.$$

It is well-known [9] that for any graph $G$, $\chi(G)-1 < \chi_c(G) \leq \chi(G)$. The question as which graphs $G$ satisfy the equality $\chi_c(G) = \chi(G)$ has received considerable attention.

Given positive integers $n \geq 2k$, the *Kneser graph* $\mathrm{KG}(n,k)$ has vertex set $\binom{[n]}{k}$, i.e., all $k$-subsets of $[n] = \{1,2,\ldots,n\}$, in which two vertices $A$ and $B$ are adjacent if $A \cap B = \emptyset$. Coloring of Kneser graphs has been a fascinating subject in graph theory. In proving Kneser's conjecture that $\chi(\mathrm{KG}(n,k)) = n-2k+2$, Lovász [6] initiated the application of algebraic topology to graph coloring. Since then, this method has became an important tool with wide applications in combinatorics.

Johnson, Holroyd and Stahl [5] first studied the circular chromatic number of Kneser graphs, and conjectured that the equality $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ always holds. This conjecture has received a lot of attention. Hajiabolhassan and Zhu [4] proved that for a fixed $k$, if $n$ is sufficiently large, then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. Meunier [7] and Simonyi and Tardos [8] proved independently that if $n$ is even then $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. The proof in [4] is combinatorial, and the proofs in [7, 8] use Fan's Lemma from algebraic topology. Nevertheless, both proofs also apply to Schrijver graphs $\mathrm{SG}(n,k)$ (subgraphs of $\mathrm{KG}(n,K)$ induced by stable

*Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan. Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan. National Center for Theoretical Sciences, Taipei Office, Taiwan. Supported in part by the National Science Council under grant NSC98-2115-M-002-013-MY3. E-mail:gjchang@math.ntu.edu.tw.

†Corresponding author. Department of Mathematics, California State University, Los Angeles, U.S.A. E-mail: dliu@calstatela.edu.

‡Department of Mathematics, Zhejiang Normal University, China. Grant Number: NSF11171310 and ZJNSF Z6110786. E-mail: xudingzhu@gmail.com.

$k$-subsets as vertices). On the other hand, it is known [8] that if $n$ is odd and is not much bigger than $2k$, then $\chi_c(\mathrm{SG}(n,k)) \neq \chi(\mathrm{SG}(n,k))$. So it seemed not of much hope to apply these methods to completely prove the Johnson-Holroyd-Stahl conjecture.

However, recently Chen [1] completely proved the Johnson-Holroyd-Stahl conjecture by using Fan's Lemma in an innovative way. A key step in Chen's proof is to prove the *Alternative Kneser Coloring Lemma*. Assume $K_{q,q}$ is a complete bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$. Denote by $K^*_{q,q}$ the graph obtained from $K_{q,q}$ by deleting the edges of a perfect matching, say by deleting the edges $x_iy_i$ $(i = 1, 2, \ldots, q)$. Assume $K^*_{q,q}$ is a subgraph $G$ and $c$ is a $q$-coloring of $G$. We say $K^*_{q,q}$ is *colorful* with respect to $c$ if $c(x_i) = c(y_i)$. Observe that if $K^*_{q,q}$ is colorful with respect to a $q$-coloring $c$, then $c(x_i) \neq c(x_j)$ for $i \neq j$, and hence we may assume that $c(x_i) = c(y_i) = i$ for $i = 1, 2, \ldots, q$.

**Lemma 1 (Alternative Kneser Coloring Lemma [1])** *Any proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$ contains a colorful copy of $K^*_{n-2k+2,n-2k+2}$.*

Note that Lovász's result is equivalent to say that for every $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$, each color class is non-empty. Chen's Alternative Kneser Coloring Lemma reveals a more delicate structure of $(n-2k+2)$-colorings for $\mathrm{KG}(n,k)$. Besides its application to the determination of the circular chromatic number of Kneser graphs, the lemma is interesting by itself. For example, it provides a positive answer to a question asked in [3]: Every optimal coloring of a Kneser graph contains a subgraph $H$ such that the close neighborhood $N_H[v]$ of each vertex of $H$ uses all the colors.

Chen's proof of Lemma 1 is rather complicated. In this article, we give a shorter proof for this result. Before presenting it, for completeness, we show how Lemma 1 is used to settle the Johnson-Holroyd-Stahl conjecture. (A simple proof of this implication is also contained in [1] and [3].)

**Lemma 2** *If $G$ is $q$-colorable and every $q$-coloring of $G$ contains a colorful copy of $K^*_{q,q}$, then $\chi_c(G) = \chi(G) = q$.*

**Proof.** For a $q$-coloring $c$ of $G$, a cycle $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is called *tight* if $c(v_{i+1}) \equiv c(v_i) + 1 \pmod q$ for $i = 0, 1, \ldots, n-1$, where the indices of the vertices are modulo $n$. It is known [9] that $\chi_c(G) = q$ if and only if $G$ is $q$-colorable and every $q$-coloring of $G$ has a tight cycle. The assumption of Lemma 2 implies that every $q$-coloring $c$ of $G$ has a tight cycle. Assume a colorful copy of $K^*_{q,q}$ with respect to $c$ has partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, with $c(x_i) = c(y_i) = i$ for $i = 1, 2, \ldots, q$. If $q$ is even, then $(x_1, y_2, x_3, y_4, \ldots, x_{q-1}, y_q, x_1)$ is a tight cycle. If $q$ is odd, then $(x_1, y_2, x_3, y_4, \ldots, y_{q-1}, x_q, y_1, x_2, y_3, x_4, \ldots, x_{q-1}, y_q, x_1)$ is a tight cycle. Thus, $\chi_c(G) = q$. ■

The Johnson-Holroyd-Stahl conjecture is an immediate consequence of Lemmas 1 and 2.

## 2 Proof of Alternative Kneser Coloring Lemma

We use Fan's Lemma to prove Chen's Alternative Kneser Coloring Lemma. Let $n$ be a positive integer and let $[-1,1]^n = \{\mathbf{x} \in R^n : ||\mathbf{x}||_\infty \leq 1\}$ be the *$n$-dimensional cube*. The *barycentric subdivision* of $[-1,1]^n$, denoted by $\mathrm{sd}([-1,1]^n)$, is the simplicial complex whose vertices are

points in $[-1,1]^n$ with each coordinate $0, 1$ or $-1$. A set of vertices form a simplex if the vertices can be ordered as $v_1, v_2, \ldots, v_t$ so that for $i = 1, 2, \ldots, t-1$, if a coordinate of $v_i$ is 1 (or $-1$, respectively) then the corresponding coordinate of $v_{i+1}$ is also 1 (or $-1$, respectively). The simplicial complex $\mathrm{sd}([-1,1]^n)$ is a triangulation of $[-1,1]^n$. The boundary of $\mathrm{sd}([-1,1]^n)$, denoted by $\partial(\mathrm{sd}([-1,1]^n))$, is a triangulation of the $(n-1)$-dimensional sphere $S^{n-1}$. Each vertex in $\partial(\mathrm{sd}([-1,1]^n))$ is a vector in $\{-1,1,0\}^n \setminus \{0\}^n$. We denote such a vector by a *signed set* $X$, which is a pair $X = (X^+, X^-)$ of disjoint subsets $X^+, X^- \subseteq [n]$, defined as $X^+ = \{i\colon X(i) = 1\}$ and $X^- = \{i\colon X(i) = -1\}$. Let $|X| = |X^+| + |X^-|$. We write $X \leq Y$ if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$, and write $X < Y$ if $X \leq Y$ and $X \neq Y$. Thus a simplex in $\partial(\mathrm{sd}([-1,1]^n))$ is a sequence of signed sets $\emptyset \neq X_1 < X_2 < \cdots < X_t$.

An $n$-labeling of $\partial(\mathrm{sd}([-1,1]^n))$ is a mapping $\lambda\colon \{-1,1,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \ldots, \pm n\}$. An $n$-labeling $\lambda$ is *antipodal* if $\lambda(-X) = -\lambda(X)$ for all $X$. A *complementary edge* with respect to $\lambda$ is a pair of signed sets $X, Y$ such that $X < Y$ and $\lambda(X) = -\lambda(Y)$. A simplex $X_1 < X_2 < \cdots < X_n$ is a *positive alternating* $(n-1)$*-simplex* with respect to $\lambda$ if $\{\lambda(X_1), \lambda(X_2), \ldots, \lambda(X_n)\} = \{1, -2, \ldots, (-1)^{n-1}n\}$. The following is a special case of Fan's Lemma.

**Octahedral Fan's Lemma** [2] *If $\lambda$ is an antipodal $n$-labeling of the vertices of $\partial(\mathrm{sd}([-1,1]^n))$ without complementary edges, then the number of positive alternating $(n-1)$-simplices is odd.*

To apply Fan's Lemma, we shall associate to each proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$ with a labeling for the vertices of $\partial(\mathrm{sd}([-1,1]^n))$. Chen's proof of the Alternative Kneser Coloring Lemma also uses this approach. The difference between the two proofs is the labelings associated to the colorings of $\mathrm{KG}(n,k)$. Chen's labeling is the composition of two functions, including a rather complicated one, while the labeling we use is direct and simple.

Assume $c$ is a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$, using colors from the set $\{2k-1, 2k, \ldots, n\}$. For a subset $S$ of $[n]$ with $|S| \geq k$, let

$$c(S) = \max\{c(A)\colon A \subseteq S, |A| = k\}.$$

Let $\prec$ be an arbitrary linear ordering on subsets of $[n]$ such that $X \prec Y$ implies $|X| \leq |Y|$. Let $\lambda\colon \{-1,1,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \ldots, \pm n\}$ be defined as follows:

$$\lambda(X) = \begin{cases} |X|, & \text{if } |X| \leq 2k-2 \text{ and } X^- \prec X^+; \\ -|X|, & \text{if } |X| \leq 2k-2 \text{ and } X^+ \prec X^-; \\ c(X^+), & \text{if } |X| \geq 2k-1 \text{ and } X^- \prec X^+; \\ -c(X^-), & \text{if } |X| \geq 2k-1 \text{ and } X^+ \prec X^-. \end{cases}$$

It is obvious that $\lambda$ is antipodal. It is also easy to verify that there are no complementary edges. Indeed, if $X < Y$ and $\lambda(X) = -\lambda(Y)$, then by definition of $\lambda$, it must be the case that $|X|, |Y| \geq 2k-1$. Assume $\lambda(X) > 0$ (the other case is symmetric). Then there exist $X' \subseteq X^+ \subseteq Y^+$ and $Y' \subseteq Y^-$ such that $|X'| = |Y'| = k$ and $c(X') = c(Y')$. However, $Y^+ \cap Y^- = \emptyset$, implying that $X'Y'$ is an edge of $\mathrm{KG}(n,k)$, a contradiction. Thus, by Fan's Lemma, there are an odd number of positive alternating $(n-1)$-simplices.

Assume $X_1 < X_2 < \cdots < X_n$ is a positive alternating $(n-1)$-simplex with respect to $\lambda$. Since $1 \leq |X_1| < |X_2| < \cdots < |X_n| \leq n$, one has $|X_i| = i$ for $1 \leq i \leq n$.

**Claim 1.** Let $X_0 = (\emptyset, \emptyset)$. For any index $1 \leq i \leq n$, either $|X_i^+| = |X_{i-1}^+| + 1$, $X_{i-1}^- = X_i^- \prec X_i^+$ and $\lambda(X_i) > 0$, or else $|X_i^-| = |X_{i-1}^-| + 1$, $X_{i-1}^+ = X_i^+ \prec X_i^-$ and $\lambda(X_i) < 0$.

*Proof.* For $1 \le i \le 2k-2$, it follows from the definitions of $\lambda$ and the positive alternating $(n-1)$-simplices that $\lambda(X_i) = (-1)^{i-1}i$, and hence if $i$ is odd, then $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^- \prec X_i^+$; if $i$ is even, then $|X_i^-| = |X_{i-1}^-| + 1$, $X_{i-1}^+ = X_i^+ \prec X_i^-$. In particular, $|X_{2k-2}^+| = |X_{2k-2}^-| = k-1$.

Assume $2k-1 \le i \le n$. Since $X_{i-1} < X_i$ and $|X_i| = |X_{i-1}| + 1$, we know that either $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^-$, or else $|X_i^-| = |X_{i-1}^-| + 1$ and $X_{i-1}^+ = X_i^+$. Assume $|X_i^+| = |X_{i-1}^+| + 1$ and $X_{i-1}^- = X_i^-$ (the other case is symmetric). Assume to the contrary of the claim that $X_i^+ \prec X_i^-$. Then $|X_i^+| \le |X_i^-|$ and so $|X_{i-1}^+| < |X_{i-1}^-|$ which gives $X_{i-1}^+ \prec X_{i-1}^-$ and $i-1 \ne 2k-2$. Hence, $\lambda(X_i) = -c(X_i^-) = -c(X_{i-1}^-) = \lambda(X_{i-1})$, contradicting the fact that $\lambda(X_r) \ne \lambda(X_s)$ for $r \ne s$. □

Since $\lceil n/2 \rceil$ of the labels $\lambda(X_i)$'s are positive and $\lfloor n/2 \rfloor$ of them are negative, it follows from Claim 1 that $|X_n^+| = \lceil n/2 \rceil$ and $|X_n^-| = \lfloor n/2 \rfloor$.

**Claim 2.** For any index $1 \le i \le n$, it holds that $-1 \le |X_i^+| - |X_i^-| \le 1$.

*Proof.* By symmetry, it is enough to show that $|X_i^+| - |X_i^-| \le 1$. Assume to the contrary that $|X_i^+| - |X_i^-| \ge 2$ for some $i$. Since $|X_n^+| - |X_n^-| \le 1$, there is an index $j$ such that $|X_{j+1}^+| - |X_{j+1}^-| \le 1 < 2 \le |X_j^+| - |X_j^-|$. Hence $|X_{j+1}^-| = |X_j^-| + 1$. By Claim 1, $X_{j+1}^+ \prec X_{j+1}^-$ and so $|X_{j+1}^+| \le |X_{j+1}^-|$, which is impossible as $|X_j^+| - |X_j^-| \ge 2$. □

It follows from Claim 2 that $|X_{2j}^+| = |X_{2j}^-| = j$ for $1 \le j \le n/2$. So we may denote $[n] = \{a_1, a_2, \ldots, a_n\}$ where $X_{2j}^+ = \{a_1, a_3, \ldots, a_{2j-1}\}$ and $X_{2j}^- = \{a_2, a_4, \ldots, a_{2j}\}$. The signed set $X_{2j-1}$ can be either $(X_{2j}^+, X_{2j-2}^-)$ or $(X_{2j-2}^+, X_{2j}^-)$.

As observed above, $\lambda(X_i) = (-1)^{i-1}i$ for $1 \le i \le 2k-2$. For $2k-1 \le i \le n$, since $\{\lambda(X_{2k-1}), \lambda(X_{2k}), \ldots, \lambda(X_n)\} = \{2k-1, -2k, \ldots, (-1)^{n-1}n\}$, by the monotonicity of $c$,

$$c(\{a_1, a_3, \ldots, a_i\}) = i \text{ for odd } i; \text{ and } c(\{a_2, a_4, \ldots, a_i\}) = i \text{ for even } i.$$

Let $\Gamma = \{X \in \{+,-,0\}^n : |X^+| = |X^-| = k-1\}$. As noted above, each positive alternating $(n-1)$-simplex contains exactly one vertex in $\Gamma$. For $X \in \Gamma$, let $\alpha(X, \lambda)$ be the number of positive alternating $(n-1)$-simplices containing vertex $X$. By Fan's Lemma, $\Sigma_{X \in \Gamma} \alpha(X, \lambda)$ is odd. Hence there exists $Z \in \Gamma$ such that $\alpha(Z, \lambda)$ is odd. In particular, there exists a positive alternating $(n-1)$-simplex $X_1 < X_2 < \cdots < X_n$ with respect to $\lambda$, with $Z = X_{2k-2}$. For this $Z$, define $\lambda' : \{+,-,0\}^n \setminus \{0\}^n \to \{\pm 1, \pm 2, \ldots, \pm n\}$ by:

$$\lambda'(X) = \begin{cases} -\lambda(X), & \text{if } X \in \{Z, -Z\}; \\ \lambda(X), & \text{otherwise.} \end{cases}$$

Then $\lambda'$ is antipodal without complementary edges. By Fan's Lemma, there are an odd number of positive alternating $(n-1)$-simplices with respect to $\lambda'$. Since $\alpha(X, \lambda') = \alpha(X, \lambda)$ for $X \in \Gamma \setminus \{Z, -Z\}$, we conclude that

$$\alpha(Z, \lambda) + \alpha(-Z, \lambda) \equiv \alpha(Z, \lambda') + \alpha(-Z, \lambda') \pmod 2.$$

Since $\lambda(Z) = -(2k-2)$ and so $\lambda(-Z) = 2k-2 = \lambda'(Z)$, we know that $\alpha(-Z, \lambda) = \alpha(Z, \lambda') = 0$. Thus, $\alpha(-Z, \lambda') \equiv \alpha(Z, \lambda) \equiv 1 \pmod 2$. So there exists a positive alternating $(n-1)$-simplex $Y_1 < Y_2 < \cdots < Y_n$ with respect to $\lambda'$, where $Y_{2k-2} = -Z$. Similar to the discussion for $\lambda$, we

may denote $[n] = \{b_1, b_2, \ldots, b_n\}$ where $Y_{2j}^+ = \{b_1, b_3, \ldots, b_{2j-1}\}$ and $Y_{2j}^- = \{b_2, b_4, \ldots, b_{2j}\}$. The signed set $Y_{2j-1}$ can be either $(Y_{2j}^+, Y_{2j-2}^-)$ or $(Y_{2j-2}^+, Y_{2j}^-)$, where $Y_0^+ = Y_0^- = \emptyset$. Also, for $2k-1 \le i \le n$, $c(\{b_1, b_3, \ldots, b_i\}) = i$ for odd $i$; and $c(\{b_2, b_4, \ldots, b_i\}) = i$ for even $i$.

Let $Z = (S, T)$. Then $X_{2k-2} = (S, T)$ and $Y_{2k-2} = (T, S)$. Consequently, for $2k-1 \le i \le n$,

$$c(S \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_i\}) = c(T \cup \{b_{2k-1}, b_{2k+1}, \ldots, b_i\}) = i \text{ for odd } i; \text{ and}$$
$$c(T \cup \{a_{2k}, a_{2k+2}, \ldots, a_i\}) = c(S \cup \{b_{2k}, b_{2k+2}, \ldots, b_i\}) = i \text{ for even } i.$$

**Claim 3.** For any index $2k-1 \le i \le n$, it holds that $a_i = b_i$ and $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$.

*Proof.* We prove by induction on $i$. If $i = 2k-1$, since $c(S \cup \{a_{2k-1}\}) = c(T \cup \{b_{2k-1}\}) = 2k-1$, so $S \cup \{a_{2k-1}\}$ and $T \cup \{b_{2k-1}\}$ are not adjacent, implying $a_{2k-1} = b_{2k-1}$. Assume $i \ge 2k$ and the claim is true for $i' < i$. If $i$ is odd, then since for all $2k-1 \le j < i$, $S \cup \{a_i\}$ and $T \cup \{a_j\}$ are adjacent, so $c(S \cup \{a_i\}) \ne c(T \cup \{a_j\}) = j$ for $2k-1 \le j < i$. Because $c(S \cup \{a_i\}) \le c(S \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_i\}) = i$, we conclude that $c(S \cup \{a_i\}) = i$. Similarly, $c(T \cup \{b_i\}) = i$. As $c(S \cup \{a_i\}) = c(T \cup \{b_i\})$, so $S \cup \{a_i\}$ and $T \cup \{b_i\}$ are not adjacent. Hence $a_i = b_i$. If $i$ is even, by the same argument, we have $c(S \cup \{b_i\}) = c(T \cup \{a_i\}) = i$, which implies that $a_i = b_i$. This completes the proof of the claim. □

The subgraph of $\mathrm{KG}(n, k)$ induced by the vertices $\{S \cup \{a_i\}, T \cup \{a_i\} : 2k-1 \le i \le n\}$ is a colorful copy of $K^*_{n-2k+2, n-2k+2}$. This completes the proof of Chen's Alternative Kneser Coloring Lemma.

**Acknowledgment.** We are grateful to the two anonymous referees for their quick reports with constructive suggestions.

## References

[1] P.-A. Chen. A new coloring theorem of Kneser graphs. *J. Combin. Theory Ser. A*, 118(3):1062–1071, 2011.

[2] K. Fan. A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math. (2)*, 56:431–437, 1952.

[3] H. Hajiabolhassan and A. Taherkhani. Graph powers and graph homomorphisms. *Electron. J. Combin.*, 17, no. 1, Research Paper 17, 16 pp, 2010.

[4] H. Hajiabolhassan and X. Zhu. Circular chromatic number of Kneser graphs. *J. Combin. Theory Ser. B*, 88(2):299–303, 2003.

[5] A. Johnson, F. C. Holroyd and S. Stahl. Multichromatic numbers, star chromatic numbers and Kneser graphs. *J. Graph Theory*, 26(3):137–145, 1997.

[6] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[7] F. Meunier. A topological lower bound for the circular chromatic number of Schrijver graphs. *J. Graph Theory*, 49(4):257–261, 2005.

[8] G. Simonyi and G. Tardos. Local chromatic number, Ky Fan's theorem and circular colorings. *Combinatorica*, 26(5):587–626, 2006.

[9] X. Zhu. Circular chromatic number: a survey. *Discrete Math.*, 229(1-3):371–410, 2001.