$L(j,k)$-Labelling and Maximum Ordering-Degrees for Trees

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Abstract

Let G be a graph. For two vertices u and v in G, we denote $d(u,v)$ the distance between u and v. Let j,k be positive integers with $j \geqslant k$. An $L(j,k)$-*labelling* for G is a function $f : V(G) \to \{0,1,2,\cdots\}$ such that for any two vertices u and v, $|f(u)-f(v)|$ is at least j if $d(u,v)=1$; and is at least k if $d(u,v)=2$. The *span* of f is the difference between the largest and the smallest numbers in $f(V)$. The $\lambda_{j,k}$-*number* for G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labellings of G. We introduce a new parameter for a tree T, namely, the maximum ordering-degree, denoted by $\mathcal{M}(T)$. Combining this new parameter and the special family of infinite trees introduced by Chang and Lu [3], we present upper and lower bounds for $\lambda_{j,k}(T)$ in terms of j, k, $\mathcal{M}(T)$, and $\Delta(T)$ (the maximum degree of T). For a special case when $j \geqslant \Delta(T)k$, the upper and the lower bounds are k apart. Moreover, we completely determine $\lambda_{j,k}(T)$ for trees T with $j \geqslant \mathcal{M}(T)k$.

1 Introduction

Motivated by the channel assignment problem (cf. Hale [14]), distance-two labelling was introduced and formulated by Griggs and Yeh [13]. Suppose a number of transmitters or stations are given. We ought to assign to each transmitter with a channel, which is a non-negative integer, such that the interference is avoided. In order to reduce the interference, any pair of 'close' transmitters must receive different channels,

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and any pair of 'very close' transmitters (between which stronger interference might occur) must receive channels that are at least two apart. The objective is to find a valid assignment such that the span of the channels used is minimized.

Graphs are natural models for the above mentioned channel assignment problem. We represent each transmitter by a vertex, where each pair of very close transmitters are connected by an edge between the corresponding vertices. A pair of close transmitters are represented by vertices of distance two apart in the graph. Hence, an $L(2,1)$-labelling is defined as follows. For a given graph G, an $L(2,1)$-*labelling* of G is a function $f : V(G) \to \{0, 1, 2, 3, \ldots\}$ such that the following hold, where $d(u, v)$ is the distance between u and v in G:

$$|f(u) - f(v)| \geqslant \begin{cases} 2, & \text{if } \ d(u,v) = 1; \\ 1, & \text{if } \ d(u,v) = 2. \end{cases}$$

The *span* of f, denoted by $\mathrm{sp}(f)$, is the difference of the largest and the smallest labels assigned to vertices, that is, $\mathrm{sp}(f) = \max\{f(V)\} - \min\{f(V)\}$. The λ-*number* of G, denoted by $\lambda(G)$, is the minimum span over all $L(2,1)$-labellings for G. Since Griggs and Yeh's first paper [13], $L(2,1)$-labelling has been studied extensively (cf. most of the References).

For a more general setting, Griggs and Yeh [13] proposed the study of a labelling f such that $|f(u) - f(v)| \geqslant m_i$ if $d(u, v) = i$ for $1 \leqslant i \leqslant N$, where N is a positive integer and $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_N \geqslant 0$ are given numbers. If $N = m_1 = 1$, f is the same as an ordinary vertex-coloring. If $N = 2$ and $m_1 \geqslant m_2 \geqslant 1$ are integers, then f is called an $L(m_1, m_2)$-labelling. That is, for given integers $j \geqslant k \geqslant 1$, the $L(j,k)$-*labelling* of G is a function f on $V(G)$ such that the following hold:

$$|f(u) - f(v)| \geqslant \begin{cases} j, & \text{if } \ d(u,v) = 1; \\ k, & \text{if } \ d(u,v) = 2. \end{cases}$$

The span of f is defined the same as an $L(2,1)$-labelling. The $\lambda_{j,k}$-*number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labellings for G. An $L(j,k)$-labelling f is called *optimal* if $\mathrm{sp}(f) = \lambda_{j,k}(G)$, and in this case f is also called a $\lambda_{j,k}$-*labelling*. For the case $k = 1$, $\lambda_{j,1}(G)$ is denoted by $\lambda_j(G)$.

For a graph G, let $\Delta(G)$ denote the maximum degree of G. Georges and Mauro [8] studied $L(j,k)$-labellings for general values of j and k. Among other results shown in [8], there are bounds of $\lambda_{j,k}(G)$ for all graphs G, in terms of i, j and

$\Delta(G)$, complete solutions (for all $j \geqslant k \geqslant 1$) of $\lambda_{j,k}(G)$ for special families of graphs including paths and cycles, and partial solutions on the products of paths. Molloy and Salavatipour [17] gave an upper bound of $\lambda_{j,k}(G)$ for planar graphs G. The λ_j-number was investigated by Chang el al. [5]. For more results on $L(j,k)$-labelling of graphs, the reader is referred to the survey articles [1, 19].

In [9], Georges and Mauro derived the $\lambda_{j,k}$-number for the infinite regular tree. In addition, for $x = j/k$, the authors introduced a rational variation $\lambda_x(G)$ of $\lambda_{j,k}(G)$, and proved that $\lambda_x(G)$ is a continuous function, for a fixed graph G. Recently, an even more general distance two labelling using real numbers as labels was introduced and studied by Griggs and Jin [11, 12].

For a graph G with maximum degree $\Delta(G)$, it is clear that $\lambda_{j,k}(G) \geqslant j + (\Delta(G) - 1)k$. The graphs achieving this bound are called $\lambda_{j,k}$-*minimal*. Chang and Lu [3] studied the structure of $\lambda_{j,k}$-minimal graphs. Using a special family of infinite trees, the authors characterized the $\lambda_{j,k}$-minimal trees [3].

The aim of this article is to investigate the $\lambda_{j,k}$-number for trees T in general. In Section 2, we introduce a new parameter for trees called the *maximum ordering-degree*, denoted as $\mathcal{M}(T)$. Combining this new parameter with the special family of infinite trees introduced in [3], in Section 2, we establish a general upper bound of $\lambda_{j,k}(T)$ for trees, in terms of i, j, and $\mathcal{M}(T)$. In Section 3 we also use $\mathcal{M}(T)$ to prove a key lemma which is utilized, in Section 4, to show a lower bound of $\lambda_{j,k}(T)$ in terms of $\mathcal{M}(T)$, provided $j \geqslant \Delta(T)k$. Moreover, we give complete solutions of $\lambda_{j,k}(T)$ for trees with $j \geqslant \mathcal{M}(T)k$.

We make a note here about the complexity problem of determining the $\lambda_{j,k}$-number for trees. Chang and Kuo [4] proved that for a tree T with maximum degree $\Delta(T)$, $\lambda(T) \in \{\Delta(T) + 1, \Delta(T) + 2\}$, and there is a polynomial-time algorithm to determine the exact value of $\lambda(T)$. Later on, an extended and similar result for $\lambda_{j,1}(T)$, $j \geqslant 1$, was shown by Chang et al. [4]. The problem, however, becomes more complicated for $k > 1$. Fiala et al. [6] showed that determining the value of $\lambda_{j,k}(T)$ is NP-complete unless j is a multiple of k, in which it can be solved polynomially.

Throughout the article, we denote T as a tree with maximum degree Δ, unless otherwise indicated.

2 Upper Bound for Trees

We introduce the maximum ordering-degree for trees and the infinite tree used by Chang and Lu [3]. Combining these two notions, we then derive an upper bound of the $\lambda_{j,k}$-number for all trees (Theorem 7).

For the special case $k = 1$, the known upper and lower bounds for the λ_j-number of trees are due to Griggs and Yeh [13] for $j = 2$, and due to Chang et al. [5] for $j \geqslant 2$.

Theorem 1 *[5] Let T be a tree with maximum degree Δ. Then $\Delta + j - 1 \leqslant \lambda_j(T) \leqslant \min\{2j + \Delta - 2, 2\Delta + j - 2\}$. Moreover, both the lower and the upper bounds are attainable.*

For the general case $1 \leqslant k \leqslant j$, the following property, observed in [3], follows naturally from the definition.

Proposition 2 *[3] For any positive integers $j \geqslant k$ and any graph G of maximum degree Δ, we have $\lambda_{j,k}(G) \geqslant j + (\Delta - 1)k$. Moreover, if the equality holds and $j > k$, then for any $\lambda_{j,k}(G)$-labelling of G, each vertex with degree Δ must be labelled by 0 or $j + (\Delta - 1)k$.*

A *$\lambda_{j,k}$-minimal tree* is a tree T with $\lambda_{j,k}(T) = j + (\Delta - 1)k$. The infinite tree introduced in [3] is defined as follows. For any positive integer M, an *$[M]$-sequence* is a sequence $(b_0, b_1, \ldots, b_t)$ for some $t \geqslant 0$, so that all the following hold:

(S1) $b_0 = 0$.

(S2) $0 \leqslant b_i \leqslant M - 1$ for all $i = 1, 2, \ldots, t$.

(S3) $b_i \geqslant b_{i-1}$ and $b_i \geqslant b_{i+1}$ for all odd $i \in \{1, 2, \ldots, t\}$.

(S4) $b_i \neq b_{i+2}$ for all $i = 0, 1, \ldots, t - 2$.

For a positive integer M, the infinite tree $\mathcal{T}_M$ has the vertex set of all $[M]$-sequences where two vertices $(b_0, b_1, \ldots, b_t)$ and $(c_0, c_1, \ldots, c_{t'})$ are adjacent if $|t - t'| = 1$ and $b_i = c_i$ for $0 \leqslant i \leqslant \min\{t, t'\}$.

Below is a characterization of the $\lambda_{j,k}$-minimal trees for large values of j.

Theorem 3 [3] *Let T be a tree of maximum degree Δ. If j, k are integers with $j \geqslant \Delta k$, then T is $\lambda_{j,k}$-minimal if and only if T is a subtree of $\mathcal{T}_\Delta$.*

The following result was proved implicitly in [3].

Theorem 4 [3] *For any positive integer M and $j \geqslant k$, $\Delta(\mathcal{T}_M) = M$ and $\lambda_{j,k}(\mathcal{T}_M) = j + (M-1)k$. Consequently, $\mathcal{T}_M$ is $\lambda_{j,k}$-minimal for any $j \geqslant k$.*

In the following, we introduce the new parameter maximum ordering-degree of a tree T. Let v be a vertex in T of degree n. The *neighborhood* of v, denoted by $\mathrm{N}(v)$, is the set of all vertices adjacent to v. The set $\mathrm{N}(v) \cup \{v\}$ is denoted by $\mathrm{N}[v]$. The *neighborhood degree sequence* of v, called ND-*sequence* for brevity, is a non-increasing degree sequence $(h_0(v), h_1(v), \ldots, h_{n-1}(v))$ of $\mathrm{N}(v)$. That is, we line up the vertices of $\mathrm{N}(v)$ by $(v_0, v_1, v_2, \cdots, v_{n-1})$ where $d(v_0) \geqslant d(v_1) \geqslant \cdots \geqslant d(v_{n-1})$, and record their degrees into the ND-sequence with $d(v_i) = h_i(v)$. We will simply use $(h_0, h_1, \cdots, h_{n-1})$ when v is understood in the context. The *maximum ordering-degree of* v, denoted by $m(v)$, is defined as

$$m(v) = \max\{h_i + i \mid 0 \leqslant i \leqslant d(v) - 1\}.$$

The *maximum ordering-degree of* T, denoted by $\mathcal{M}(T)$, is defined as

$$\mathcal{M}(T) = \max\{m(v) \mid v \in V(T)\}.$$

For instance, the maximum ordering-degree for $K_{1,\Delta}$ (a star with Δ leaves) is Δ.

Throughout the article, we shall simply denote $\mathcal{M}(T)$ by $\mathcal{M}$, when T is clear in the context.

Lemma 5 *Let T be a tree and v a vertex in T with degree n. Let $(d_0, d_1, d_2, \cdots, d_{n-1})$ be any ordering of the degrees of* $\mathrm{N}(v)$ *and let $(h_0, h_1, \cdots, h_{n-1})$ be the* ND-*sequence of v. Then* $\max\{d_i + i \mid 0 \leqslant i \leqslant n-1\} \geqslant m(v)$.

Proof. Let $m(v) = h_j + j$ for some $0 \leqslant j \leqslant n-1$. Since $\{d_0, d_1, d_2, \ldots, d_{n-1}\} = \{h_0, h_1, \cdots, h_{n-1}\}$ and $h_0 \geqslant h_1 \geqslant h_2 \geqslant \ldots \geqslant h_{n-1}$, there exists some $t \geqslant j$ such that $h_j \leqslant d_t$. Hence, $\max\{d_i + i \mid 0 \leqslant i \leqslant n-1\} \geqslant d_t + t \geqslant h_j + j = m(v)$. □

The next result emerges directly from the definition of $\mathcal{M}(T)$.

Lemma 6 *Let T be a tree with maximum degree Δ and maximum ordering-degree $\mathcal{M}$. Then $\Delta \leqslant \mathcal{M} \leqslant 2\Delta - 1$.*

For two graphs G and H, a graph *homomorphism* from G to H is an edge-preserving function from $V(G)$ to $V(H)$. If there exists an injective homomorphism from G to H, then by composition of functions, one gets $\lambda_{j,k}(G) \leqslant \lambda_{j,k}(H)$.

We now show a general upper bound of the $\lambda_{j,k}$-number for trees.

Theorem 7 *Let T be a tree with maximum ordering-degree $\mathcal{M}$. Let j, k be integers with $j \geqslant k$. Then $\lambda_{j,k}(T) \leqslant j + \mathcal{M}k$.*

Proof. By Theorem 4, it suffices to find an injective homomorphism from $V(T)$ to $V(\mathcal{T}_{\mathcal{M}+1})$. Let v be a vertex with $m(v) = \mathcal{M}$. Suppose v has degree n. Let $N(v) = \{v_0, v_1, \cdots, v_{n-1}\}$ with ND-sequence $(h_0, h_1, \cdots, h_{n-1})$, where h_i is the degree of v_i. We define a homomorphism by the following recursive process which labels each vertex in $V(T)$ with an $[\mathcal{M}+1]$-sequence.

Initially, all the vertices are unlabelled. Label v by (0) and its neighbors v_i by $(0, \mathcal{M} - 1 - i)$, $0 \leqslant i \leqslant n-1$. Suppose a vertex u has been labelled by $(b_0, b_1, \cdots, b_t)$ for some $t \geqslant 1$. We next label all the unlabelled neighbors of u (if there exists any). Note, by our labelling scheme, there is only one neighbor of u that has been labelled. Let $\{u_i \mid 0 \leqslant i \leqslant d(u) - 2\}$ be the set of unlabelled neighbors of u, where $d(u_i) = d_i$ and $d_i \geqslant d_{i+1}$. Observe, $d_i \leqslant h_i(u)$. We label u_i by $(b_0, b_1, \cdots, b_t, b_{t+1}^{(i)})$ according to the following:

If t is even, let

$$b_{t+1}^{(i)} = \begin{cases} \mathcal{M} - i, & \text{if } \mathcal{M} - i \geqslant b_{t-1} + 1; \\ \mathcal{M} - i - 1, & \text{otherwise.} \end{cases}$$

So $b_{t+1}^{(i)} \neq b_{t-1}$ and $b_{t+1}^{(i)} \leqslant \mathcal{M}$, if t is even.

If t is odd, let

$$b_{t+1}^{(i)} = \begin{cases} i, & \text{if } i \leqslant b_{t-1} - 1; \\ i + 1, & \text{otherwise.} \end{cases}$$

So $b_{t+1}^{(i)} \neq b_{t-1}$ and $b_{t+1}^{(i)} \leqslant i + 1 \leqslant d(u) - 1 \leqslant \Delta(T) - 1 \leqslant \mathcal{M} - 1$, if t is odd.

Clearly, all the labels satisfy conditions (S1), (S2) and (S4) for an $[\mathcal{M}+1]$-sequence. We check (S3). As $d_i \leqslant h_i(u)$, we have

$$\max\{d_i + i\} \leqslant \max\{h_i(u) + i\} \leqslant m(u) \leqslant \mathcal{M}.$$

Hence, $d_i \leqslant \mathcal{M} - i$ for every i. Let w be a vertex adjacent to u_i and labelled by $(b_0, b_1, \cdots, b_t, b_{t+1}^{(i)}, b_{t+2})$. If t is even, then $b_{t+2} \leqslant d_i - 1 \leqslant \mathcal{M} - i - 1 \leqslant b_{t+1}^{(i)}$. If t is odd, then $b_{t+2} \geqslant \mathcal{M} - 1 - (d_i - 2) \geqslant i + 1 \geqslant b_{t+1}^{(i)}$. This verifies (S3). Hence, the mapping by the labelling is well-defined. Moreover, by definition it is clear that the mapping is injective and edge-preserving. This completes the proof. □

3 Key Lemma

The aim of this section is to establish Lemma 10, which will be used in the next section. Let T be a tree with maximum degree Δ and maximum ordering-degree $\mathcal{M}(T) > \Delta$. Let v be a vertex in T with $m(v) > \Delta$ and let the ND-sequence of v be $(h_0, h_1, \cdots, h_{n-1})$, where $n = \deg(v)$. In the following we define a recursive operation, by first lining up the neighbors of v by $\mathrm{N}(v) = (v_0, v_1, v_2, \cdots, v_{n-1})$ where $d(v_i) = h_i$.

The recursive process begins with two lists, $(v_0, v_1, \cdots, v_{n-1})$ and $(h_0, h_1, \cdots, h_{n-1})$. In each step we delete one vertex v_t and its corresponding h_t from the lists, where the value of t is determined by the function σ introduced below. The new list of h_i's remains in non-increasing order.

Here is the precise definition of the process. Initially, let $v_i^{(0)} = v_i$ and $h_i^{(0)} = h_i$ for $0 \leqslant i \leqslant n-1$, and let $m^{(0)}(v) = m(v)$.

For $q \geqslant 1$, if $m^{(q-1)}(v) \leqslant \Delta$ then stop; else $m^{(q-1)}(v) > \Delta$ then define

$$\sigma(q) = \min\{i \mid h_i^{(q-1)} + i = m^{(q-1)}(v), 0 \leqslant i \leqslant n-q\}, \text{ and}$$

$$\begin{cases} v^{(q)} = v_{\sigma(q)}^{(q-1)} \text{ and } h^{(q)} = h_{\sigma(q)}^{(q-1)}, \\ v_i^{(q)} = v_i^{(q-1)} \text{ and } h_i^{(q)} = h_i^{(q-1)}, & \text{if } 0 \leqslant i \leqslant \sigma(q) - 1, \\ v_i^{(q)} = v_{i+1}^{(q-1)} \text{ and } h_i^{(q)} = h_{i+1}^{(q-1)}, & \text{if } \sigma(q) \leqslant i \leqslant n-q-1, \\ m^{(q)}(v) = \max\{h_i^{(q)} + i \mid 0 \leqslant i \leqslant n-q-1\}. \end{cases}$$

In the following, we remark some properties for the above recursive process. In particular, we show that the process stops at some point.

Proposition 8 *Let T be a tree with maximum degree $\Delta < \mathcal{M}(T)$. Let v be a vertex of degree n and $m(v) > \Delta$. Let $p = m(v) - \Delta$. For the process defined above, the following hold for any $1 \leqslant q \leqslant p$.*

(1) $\sigma(q) \geqslant 1$.

(2) $h_0^{(q)}, h_2^{(q)}, \ldots, h_{n-q-1}^{(q)}$ *is a non-increasing sequence.*

(3) $h^{(q-1)}_{\sigma(q)-1} = h^{(q-1)}_{\sigma(q)}$.

(4) $m^{(q)}(v) = m^{(q-1)}(v) - 1 = m(v) - q$. *Consequently, the process stops at* $q = m(v) - \Delta = p$.

(5) $\sigma(q) < \sigma(q-1)$. *Consequently,* $h^{(q)} \geqslant h^{(q-1)}$ *for* $2 \leqslant q \leqslant p$.

(6) $v_i^{(q)} = v_i$ *for all* $0 \leq i \leq \sigma(q) - 1$.

Proof. To show (1), suppose $\sigma(q) = 0$ for some $1 \leqslant q \leqslant p-1$. Then by definition, $m^{(q-1)}(v) = h_0^{(q-1)} + 0 = h_0^{(q-1)} \leqslant \Delta$, a contradiction.

(2) follows since initially $(h_0, h_1, \cdots, h_{n-1})$ is in a non-increasing order, and the ordering is kept in each step.

To show (3), by (1) and (2) we get $h^{(q-1)}_{\sigma(q)-1} \geqslant h^{(q-1)}_{\sigma(q)}$. Assume to the contrary, $h^{(q-1)}_{\sigma(q)-1} > h^{(q-1)}_{\sigma(q)}$. Then $h^{(q-1)}_{\sigma(q)-1} + \sigma(q) - 1 \geqslant h^{(q-1)}_{\sigma(q)} + \sigma(q)$, contradicting the choice of $\sigma(q)$.

To show (4), let $m^{(q-1)}(v) = x$ and $m^{(q)}(v) = x'$. By definition, $x' \leqslant x - 1$. By (3), $x' \geqslant x - 1$. Finally, (5) follows by (3) and (4); (6) follows by (5) and the definition of $v^{(q)}$. □

The next result follows directly from the definition of an $L(j,k)$-labelling.

Lemma 9 *Let* v *be a degree* n *vertex in* G. *If* f *is an* $L(j,k)$*-labelling for* G *such that* $f(w) < f(v) < f(u)$ *for some* $w, u \in \mathrm{N}(v)$, *then* $\mathrm{sp}(f) \geqslant 2j + (n-2)k$.

Lemma 10 *Let* T *be a tree with maximum degree* Δ. *Suppose* v *is a vertex of* T *with degree* n, $m(v) = \mathcal{M}(T)$, *and* $d(v,u) \leqslant 2$ *for all vertex* $u \neq v$ *in* T. *Let the* ND*-sequence of* v *be* $(h_0, h_1, h_2, \cdots, h_{n-1})$, $p = m(v) - \Delta$, *and let* $h^{(q)}$, $1 \leqslant q \leqslant p$, *be defined as in the above process. Then all the following hold.*

(a) $\lambda_{j,k}(T) \leqslant j + (m(v)-1)k$ *for all* $j \geqslant k$.

(b) If $m(v) = \Delta$, *then* $\lambda_{j,k}(T) = j + (\Delta - 1)k$.

(c) If $m(v) > \Delta$, *then* $\lambda_{j,k}(T) = j + (m(v)-1)k$ *if* $j \geqslant \Delta k$; *otherwise,*

$$\lambda_{j,k}(T) \geqslant \begin{cases} \min\{j + (m(v)-1)k, 2j + (\min\{n, h^{(1)}\} - 2)k\}, \\ \qquad \textit{if } (m(v) - h^{(1)})k \leqslant j < \Delta k; \\ \min\{j + (m(v)-q-1)k, 2j + (\min\{n, h^{(q+1)}\} - 2)k\}, \\ \qquad \textit{if } (m(v) - h^{(q+1)} - q)k \leqslant j < (m(v) - h^{(q)} - q + 1)k, \\ \qquad \textit{for some } 1 \leqslant q \leqslant p-1; \\ \min\{j + (\Delta - 1)k, 2j + (n-2)k\}, \quad \textit{if} \quad j < (\Delta - h^{(p)} + 1)k. \end{cases}$$

(d) If $n \geqslant h^{(i)}$ for some $i \geqslant 1$, then the conclusion (in-equality) in (c) becomes an equality for $j \geqslant (m(v) - h^{(i)} - i + 1)k$. In particular, if $n \geqslant h^{(p)}$ then the above determines the value of $\lambda_{j,k}(T)$ for $j \geqslant (\Delta - h^{(p)} + 1)k$.

Proof. Let $\mathrm{N}(v) = \{v_0, v_1, \cdots, v_{n-1}\}$ with $d(v_i) = h_i$ and $\mathrm{N}(v_i) = \{v, u_{(i,1)}, u_{(i,2)}, \cdots, u_{(i,h_i-1)}\}$ for $0 \leqslant i \leqslant n-1$. To show (a), we give an $L(j,k)$-labelling g for T by: $g(v) = j+(m(v)-1)k$, $g(v_i) = ik$ for $0 \leqslant i \leqslant n-1$, and $g(u_{(i,d)}) = j+(m(v)-1-d)k$. It is easy to see that g is an $L(j,k)$-labelling for T.

If $m(v) = \Delta$, then (b) follows from Proposition 2 and (a).

Suppose $m(v) > \Delta$. Let f be an optimal $L(j,k)$-labelling of T. Since all the neighbors of v must receive different labels, we can line up the neighbors of v by $u_0, u_1, \cdots, u_{n-1}$ so that their labels are in an increasing order. That is,

$$f(u_i) \geqslant f(u_{i-1}) + k \quad \text{for } 1 \leqslant i \leqslant n-1.$$

Let d_i denote the degree of u_i, $0 \leqslant i \leqslant n-1$. (Note, $(d_0, d_1, \cdots, d_{n-1})$ is not necessarily an ND-sequence.) Hence, $f(u_i) \geqslant f(u_0) + ik \geqslant ik$ for all $i = 0, 1, \cdots, n-1$.

A vertex w is called *straight* if $f(w) > f(w')$ for all $w' \in \mathrm{N}(w)$, or $f(w) < f(w')$ for all $w' \in \mathrm{N}(v)$. There are three cases to consider.

Case 1. v is not straight. By Lemma 9,

$$\mathrm{sp}(f) \geqslant 2j + (n-2)k. \tag{3.1}$$

Case 2. Every $x \in \mathrm{N}[v]$ is straight. We claim:

$$\mathrm{sp}(f) \geqslant j + (m(v)-1)k. \tag{3.2}$$

By symmetry, we assume without loss of generality $f(v) > f(u_i)$ for all $0 \leqslant i \leqslant n-1$. For any $0 \leqslant i \leqslant n-1$, since u_i is straight and $v \in N(u_i)$, we get $f(u_i) < f(z)$ for all $z \in \mathrm{N}(u_i)$. For $0 \leqslant i \leqslant n-1$, let $f_i = \max\{f(z) \mid z \in \mathrm{N}(u_i)\}$. Then, $f_i \geqslant j + (d_i - 1)k + f(u_i) \geqslant j + (d_i - 1 + i)k$. So,

$$\begin{aligned} \mathrm{sp}(f) &\geqslant \max\{f_i \mid 0 \leqslant i \leqslant n-1\} \\ &\geqslant j + (\max\{d_i + i \mid 0 \leqslant i \leqslant n-1\} - 1)k \\ &\geqslant j + (m(v)-1)k. \end{aligned}$$

The last inequality follows by Lemma 5. This proves (3.2).

Case 3. v is straight, but u_i is not for some i. We claim (3.3) or (3.4) holds:

$$\mathrm{sp}(f) \geqslant \max\{2j,\ j+(m(v)-1)k\}, \tag{3.3}$$

$$\mathrm{sp}(f) \geqslant \min_{1\leqslant i\leqslant p}\left\{\max\{2j+(h^{(i)}-2)k,\ j+(m^{(i)}(v)-1)k\}\right\}. \tag{3.4}$$

Partition $\mathrm{N}(v)$ into two parts, A and S, where S consists of all the straight vertices in $\mathrm{N}(v)$, and $A = \mathrm{N}(v) - S$. Then $A \neq \emptyset$. Let $A = \{u_1, u_2, \cdots, u_x\}$ and $d(u_i) = t_i$ for $1 \leqslant i \leqslant x$. Let T_A and T_S be two induced subtrees of T where $V(T_A) = \cup_{w\in A}\mathrm{N}[w]$ and $V(T_S) = \cup_{w\in S}\mathrm{N}[w]$. Note, $v \in V(T_A) \cap V(T_S)$. Let g_1 and g_2 be the restrictions of f to $V(T_A)$ and $V(T_S)$, respectively. Then, $\mathrm{sp}(f) \geqslant \max\{\mathrm{sp}(g_1), \mathrm{sp}(g_2)\}$.

Let $h^* = \max\{t_1, t_2, \cdots, t_x\}$. By Lemma 9, $\mathrm{sp}(g_1) \geqslant 2j + (h^*-2)k$. For T_S, by the same discussion in Case 2, $\mathrm{sp}(g_2) \geqslant j+(m'(v)-1)k$, where $m'(v)$ is the maximum ordering-degree of v restricted to the subtree T_S. Hence,

$$\mathrm{sp}(f) \geqslant \max\{2j+(h^*-2)k,\ j+(m'(v)-1)k\}. \tag{3.5}$$

First, assume $m'(v) \geqslant \Delta$. Let $q = m(v) - m'(v)$. So, $0 \leqslant q \leqslant m(v) - \Delta$. If $q = 0$, as $A \neq \emptyset$, we have $h^* - 2 \geqslant 0$, hence (3.3) holds. Suppose $q \geqslant 1$. By Proposition 8 (4), $m^{(q)}(v) = m(v) - q = m'(v)$. Therefore, to prove (3.4), by (3.5) it suffices to verify the following:

$$\max\{2j+(h^*-2)k, j+(m'(v)-1)k\} \geqslant \max\{2j+(h^{(q)}-2)k, j+(m^{(q)}(v)-1)k\},$$

which is equivalent to $h^* \geqslant h^{(q)}$. Assume to the contrary, $h^* < h^{(q)}$. By Proposition 8 (5)(6), $v^{(q)} = v^{(q-1)}_{\sigma(q)} = v_{\sigma(q)}$. Hence, $h^{(q)} = h^{(q-1)}_{\sigma(q)} = h_{\sigma(q)}$. Since h^* is the largest degree in $V(T_A) - \{v\}$ and as $h^* < h^{(q)}$, we conclude that the vertices v_i with $0 \leqslant i \leqslant \sigma(q)$ are all in T_S. This implies that $m'(v) \geqslant h_{\sigma(q)} + \sigma(q) = h^{(q-1)}_{\sigma(q)} + \sigma(q) = m^{(q-1)}(v) = m^{(q)}(v) + 1 > m^{(q)}(v)$, a contradiction. Hence, (3.4) holds.

Next, assume $m'(v) < \Delta$. Since f is optimal, $\mathrm{sp}(f) \geqslant j + (\Delta-1)k$. We can replace $m'(v)$ in (3.5) by Δ. Let $q = m(v) - \Delta$. A similar proof as the above will lead to the same conclusion (where $m^{(q)}(v) = m(v) - q = \Delta > m'(v)$).

Now, to get a lower bound for $\lambda_{j,k}(T)$, it suffices to get the least bound among the ones in (3.1), (3.2), (3.3), and (3.4). Notice that (3.2) is weaker than (3.3), so we shall only consider (3.1), (3.2) and (3.4).

Assume $j \geqslant \Delta k$. Then $j + (m(v)-1)k \leqslant 2j + (h^{(1)}-2)k$, since by definition $h^{(1)} + n - 1 \geqslant h^{(1)} + \sigma(1) = m(v)$ and because $n \leqslant \Delta$. By the fact that $h^{(1)} \leqslant$

$h^{(2)} \leqslant \cdots \leqslant h^{(p)}$, we conclude that the minimum among (3.1), (3.2), and (3.4) gives $\lambda_{j,k}(T) \geqslant j + (m(v)-1)k$, for $j \geqslant \Delta k$. Hence, by (a), $\lambda_{j,k}(T) = j + (m(v)-1)k$, for $j \geqslant \Delta k$.

Assume $j < \Delta k$. By Proposition 8, we have

$$j + (m(v)-1)k > j + (m^{(1)}(v)-1)k > \cdots > j + (m^{(p)}(v)-1)k = j + (\Delta - 1)k, \text{ and}$$

$$2j + (h^{(1)} - 2)k \leqslant 2j + (h^{(2)} - 2)k \leqslant \cdots \leqslant 2j + (h^{(p)} - 2)k. \tag{3.6}$$

To find the minimum among (3.1), (3,2) and (3.4), we consider different values of j. Notice that

$$2j + (h^{(x)}(v) - 2)k \geqslant j + (m^{(x)}(v) - 1)k \iff j \geqslant (m^{(x-1)}(v) - h^{(x)}(v))k. \tag{3.7}$$

Assume $(m(v) - h^{(1)})k \leqslant j < \Delta k$. Then

$$\max\{2j + (h^{(1)} - 2)k, j + (m(v) - 2)k\} = 2j + (h^{(1)} - 2)k.$$

Therefore the least bound among (3.1), (3.2) and (3.4) gives

$$\lambda_{j,k}(T) \geqslant \min\{j + (m(v)-1)k, 2j + (h^{(1)} - 2)k, 2j + (n-2)k\}.$$

Hence, the result for the case $(m(v) - h^{(1)})k \leqslant j < \Delta k$ follows. Similarly, by (3.6) and (3.7) the remaining cases in (c) can be obtained; we should leave the details to the reader.

Now we prove (d). Assume $j \geqslant (m(v) - h^{(q)} - q + 1)k$, and $n \geqslant h^{(q)}$ for some $1 \leqslant q \leqslant p$. Then $2j + (h^{(q)} - 2)k \geqslant j + (m^{(q)} - 1)k$, and $n \geqslant h^{(x)}$ for $1 \leqslant x \leqslant q$. For any $1 \leqslant x \leqslant q$, let

$$Q_x = \max\{2j + (h^{(x)} - 2)k, j + (m^{(x)}(v) - 1)k\}.$$

To prove (d), it suffices to find $L(j,k)$-labellings g and f for T with spans Q_x and $j + (m(v)-1)k$, respectively.

Let $g(v) = Q_x$. For $0 \leqslant i \leqslant n - x - 1$, let $g(v_i^{(x)}) = ik$ and $g(u_{(i,d)}) = j + (m^{(x)}(v) - 1 - d)k$, where $u_{(i,d)} \in \mathrm{N}(v_i^{(x)}) - \{v\}$ and $1 \leqslant d \leqslant h_i^{(x)} - 1$. For $1 \leqslant i \leqslant x$, let $g(v^{(i)}) = Q_x - j - (i-1)k$ and $g(w_d^{(i)}) = Q_x - dk$ for $d \leqslant i - 1$ or $g(w_d^{(i)}) = Q_x - 2j - (d-1)k$ for $d \geqslant i$, where $w_d^{(i)} \in \mathrm{N}(v^{(i)}) - \{v\}$ and $1 \leqslant d \leqslant h^{(i)} - 1$.

Let $f(v) = j + (m(v)-1)k$. For $0 \leqslant i \leqslant n-1$, let $f(v_i) = ik$ and label the unlabelled neighbors of v_i by $j + (m(v) - 1 - l)k$, $l = 1, \cdots, h_i - 1$.

It is easy to see that both g and f are $L(j,k)$-labellings for T, with the desired spans. This completes the proof. □

4 Lower Bound and Large Values of j

We use the maximum ordering-degree of a tree T to develop a lower bound of $\lambda_{j,k}(T)$ for $j \geqslant \Delta(T)k$. Moreover, we completely determine the value of $\lambda_{j,k}(T)$ for trees T with $j \geqslant \mathcal{M}(T)k$.

Theorem 11 *Let T be a tree with maximum degree Δ and maximum ordering-degree $\mathcal{M}$. If $j \geqslant \Delta k$, then $\lambda_{j,k}(T) \geqslant j + (\mathcal{M}-1)k$.*

Proof. If $\mathcal{M} = \Delta$, the result holds by Proposition 2. Assume $\mathcal{M} > \Delta$. Let v be a vertex with degree n and $m(v) = \mathcal{M}$. Let H be the subtree of T induced by v and all the vertices within distance 2 from v. Since $j \geqslant \Delta k$, by Lemma 10, we have $\lambda_{j,k}(H) = j + (\mathcal{M}-1)k$. Hence $\lambda_{j,k}(T) \geqslant j + (\mathcal{M}-1)k$. □

By Theorems 4, 7 and 11, we obtain

Corollary 12 *Let T be a tree with maximum degree Δ and maximum ordering-degree $\mathcal{M}$. Assume j and k are integers with $j \geqslant \Delta k$. Then $j+(\mathcal{M}-1)k \leqslant \lambda_{j,k}(T) \leqslant j+\mathcal{M}k$. Moreover, if T is a subtree of $\mathcal{T}_{\mathcal{M}}$, then $\lambda_{j,k}(T) = j + (\mathcal{M}-1)k$.*

It was proved by Georges and Mauro [8] that for any graph G, $\lambda_{j,k}(G) = \alpha j + \beta k$ for some non-negative integers α and β. By Corollary 12, if T is a tree of maximum degree Δ and maximum ordering-degree $\mathcal{M}$, and if $j \geqslant \Delta k$, then there only three possible values for $\lambda_{j,k}(T)$. Precisely, if $j \geqslant \Delta k$, then

$$\lambda_{j,k}(T) \in \{j + (\mathcal{M}-1)k,\ \beta k,\ j + \mathcal{M}k\},$$

where $\mathcal{M}$ is the maximum ordering-degree of T, and β is a non-negative integer with $j + (\mathcal{M}-1)k \leqslant \beta k \leqslant j + \mathcal{M}k$.

Theorem 13 *Let j, k, M be positive integers with with $j \geqslant Mk$. For any graph G, the following are equivalent:*

(1) $\lambda_{j,k}(G) < j + Mk$.

(2) There exists a $\lambda_{j,k}$-labelling g for G such that for any vertex v in G, $g(v)$ is of the form $a_v j + b_v k$ with $a_v \in \{0, 1\}$ and $b_v \in \{0, 1, \cdots, M-1\}$. Moreover, the following hold:

(D1) *If* $d_G(u,v)=1$, *then* $a_u \neq a_v$. *If* $a_u=0$ *and* $a_v=1$, *then* $b_u \leqslant b_v$.

(D2) *If* $d_G(u,v)=2$, *then* $a_u=a_v$ *and* $b_u \neq b_v$.

(3) $\lambda_{j,k}(G) \leqslant j+(M-1)k$.

Proof. It is enough to show (1) $\Rightarrow$ (2). Assume $\lambda_{j,k}(G) < j+Mk$. Let f be a $\lambda_{j,k}$-labelling for G. For every vertex v, write $f(v)=a_vj+b_vk+r_v$, where a_v, b_v, r_v are non-negative integers satisfying $0 \leqslant b_vk+r_v < j$ and $0 \leqslant r_v < k$.

Since $j \geqslant Mk$ and $\lambda_{j,k}(G) < j+Mk$, we have $a_v=0$ or 1. We claim that $0 \leqslant b_v \leqslant M-1$. If $a_v=1$, this is obviously true. Assume $a_v=0$. Choose a vertex u adjacent to v. (If v is isolated, let $f(v)=0$.) By the assumption that $f(v)=b_vk<j$, it must be $f(v)<f(u)$, and so $f(v) \leqslant f(u)-j < Mk$, implying $b_v \leqslant M-1$.

Define a function g on $V(G)$ by $g(v)=a_vj+b_vk$. It suffices to show that g is an $L(j,k)$-labelling for G, and g satisfies (D1) and (D2). Suppose u and v are adjacent vertices in G. As $|f(u)-f(v)| \geqslant j \geqslant Mk$, it must be the case that $a_u \neq a_v$. Without loss of generality, assume $a_u=0$ and $a_v=1$. Then $f(u)<f(v)$. So we have $f(u) \leqslant f(v)-j$, implying $b_uk+r_u \leqslant b_vk+r_v$. As $b_uk \leqslant b_uk+r_u \leqslant b_vk+r_v < (b_v+1)k$, we get $b_u \leqslant b_v$. Hence, $g(u) \leqslant g(v)-j$, and (D1) holds.

Next, suppose $d_G(u,v)=2$. Choose a vertex w adjacent to both u and v. Then $a_u \neq a_w$ and $a_w \neq a_v$, implying $a_u=a_v$. As $|f(u)-f(v)| \geqslant k$, we have $b_u \neq b_v$ (so (D2) holds) and $|g(u)-g(v)| \geqslant k$. This verifies that g is a $\lambda_{j,k}$-labelling for G, which satisfies (D1) and (D2). □

Let T be a tree with maximum ordering-degree $\mathcal{M}$. Let $j \geqslant \mathcal{M}k$. Because $\mathcal{M}(T) \geqslant \Delta(T)$, by Theorem 13 and Corollary 12, we have $\lambda_{j,k}(T) \in \{j+(\mathcal{M}-1)k, j+\mathcal{M}k\}$. Indeed, the value of $\lambda_{j,k}(T)$ can be completely settled as follows.

Theorem 14 *Let* T *be a tree with maximum ordering-degree* $\mathcal{M}$. *If* $j \geqslant \mathcal{M}k$, *then*

$$\lambda_{j,k}(T)=\begin{cases} j+(\mathcal{M}-1)k, & \text{if } T \text{ is a subtree of } \mathcal{T}_{\mathcal{M}};\\ j+\mathcal{M}k, & \text{otherwise.}\end{cases}$$

Proof. Assume $j \geqslant \mathcal{M}k$. By Theorem 13 and Corollary 12, it is enough to show that if $\lambda_{j,k}(T)=j+(\mathcal{M}-1)k$, then T is a subtree of $\mathcal{T}_{\mathcal{M}}$.

Suppose $\lambda_{j,k}(T)=j+(\mathcal{M}-1)k$. By Theorem 13, T has a $\lambda_{j,k}$-labelling g such that for any vertex v, $g(v)=a_vj+b_vk$ where $a_v \in \{0,1\}$ and $b_v \in \{0,1,\cdots,M-1\}$ satisfying conditions (D1) and (D2).

Let v be a vertex with $m(v) = \mathcal{M}$. Since g is a $\lambda_{j,k}$-labelling, there exists a vertex v_0 such that $g(v_0) = 0$. To prove that T is a subtree of $\mathcal{T}_{\mathcal{M}}$, it suffices to find an injective homomorphism from T to $\mathcal{T}_{\mathcal{M}}$.

For any vertex u in T, there is a unique path from v_0 to u. Denote this path by $P : v_0, v_1, \cdots, v_m = u$. According to (D1) and (D2), $(b_{v_0}, b_{v_1}, \cdots, b_{v_m})$ is an $[\mathcal{M}]$-sequence. Define a function $\phi : V(T) \to V(\mathcal{T}_{\mathcal{M}})$ by $\phi(u) = (b_{v_0}, b_{v_1}, \cdots, b_{v_m})$. Clearly, ϕ is an injective homomorphism. Hence T is a subtree of $\mathcal{T}_{\mathcal{M}}$. □

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