THE LENGTH OF NOETHERIAN POLYNOMIAL RINGS

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Abstract. We show that if R is a left Noetherian ring, then

$$\operatorname{len} R[x] = \omega \otimes \operatorname{len} R.$$

Here, for a Noetherian left module A, $\operatorname{len} A$ is its ordinal valued length as defined by Gulliksen [1], and $\otimes$ is the natural product on ordinal numbers.

1. Introduction

One of the most studied themes of ring theory is that of the relationship between a ring R and $R[x]$, the ring of polynomials over R. One important example of this is the following: If R is a left Noetherian ring then so is $R[x]$, and the left Krull dimensions of these rings are related by the equation $\operatorname{Kdim} R[x] = \operatorname{Kdim} R + 1$. For the definitions and the proofs of these claims see [2, Ch. 13], [3] or [4, Ch. 6].

The size of a Noetherian module A is rather coarsely characterized by its Krull dimension, $\operatorname{Kdim} A$. A more precise measure is its ordinal valued length, $\operatorname{len} A$. This was first defined by Gulliksen [1], and is a generalization of both Krull dimension and length in the usual sense. Specifically, if A has finite length, then $\operatorname{len} A$ has the usual meaning, and if $\operatorname{len} A$ is written in normal form, $\operatorname{len} A = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are ordinals, then $\operatorname{Kdim} A = \gamma_1$. The advantage of the ordinal valued length over Krull dimension is seen in the following cancellation result which is trivial to prove using the ordinal length: If A, B and C are Noetherian modules such that $A \oplus C \cong B \oplus C$, then $\operatorname{len} A = \operatorname{len} B$, and, in particular, $\operatorname{Kdim} A = \operatorname{Kdim} B$. For the proof of these claims and further information about the ordinal valued length, see [1] or [5].

The main theorem (3.1) of this paper says that if R is a left Noetherian ring, then

$$\operatorname{len} R[x] = \omega \otimes (\operatorname{len} R).$$

Since the Krull dimensions of these modules are encoded in their lengths, the equation $\operatorname{Kdim} R[x] = \operatorname{Kdim} R + 1$ follows as a corollary.

For a Noetherian module A, $\operatorname{len} A$ is really a measure of the size of $\mathcal{L}^\circ(A)$, the lattice of submodules of A ordered by reverse inclusion. This lattice is Artinian, and the main theorem is a relatively easy consequence of a theorem about Artinian partially ordered sets which we prove in Section 2. Specifically, let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered sets, $\Downarrow\mathcal{K}$ the set of lower sets of $\mathcal{K}$ ordered by inclusion, and $\operatorname{dec}(\mathcal{K},\mathcal{L})$ the set of decreasing functions from $\mathcal{K}$ to $\mathcal{L}$. If $\Downarrow\mathcal{K}$ and $\mathcal{L}$ are Artinian, then $\operatorname{dec}(\mathcal{K},\mathcal{L})$ is Artinian. If, in addition $\mathcal{L}$ is a modular lattice, or $\Downarrow\mathcal{L}$ is Artinian, or $\mathcal{L}$ contains a chain of length $\operatorname{len}\mathcal{L}$, then

$$\operatorname{len}(\operatorname{dec}(\mathcal{K},\mathcal{L})) = (\operatorname{len}\Downarrow\mathcal{K}) \otimes (\operatorname{len}\mathcal{L}).$$

As an corollary we have an easier proof of an old result: If $\Downarrow\mathcal{K}$ and $\Downarrow\mathcal{L}$ are Artinian, then

$$\operatorname{len}(\Downarrow(\mathcal{K}\times\mathcal{L})) = (\operatorname{len}\Downarrow\mathcal{K}) \otimes (\operatorname{len}\Downarrow\mathcal{L}).$$

2. Artinian Partially Ordered Sets

Let **Ord** be the class of ordinal numbers, ω the smallest infinite ordinal, $\mathbb{Z}^+ = \{0,1,2,\dots\}$ and $\mathbb{N}$=$\{1,2,3,\ \dots\ \}$. Any nonzero ordinal α can be expressed uniquely in **normal form**, $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are ordinals. Since $\omega^\gamma n = \omega^\gamma + \omega^\gamma + \ldots + \omega^\gamma$ (n times) when $n \in \mathbb{N}$, it is possible to collect together terms in the normal form which have identical exponents. Thus α can also be written uniquely in the form $\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$ where now $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ and $n_1, n_2, \dots, n_n \in \mathbb{N}$.

The natural sum $\oplus$ and natural product $\otimes$ of two ordinals are defined essentially by operating on these normal forms as if they were polynomials over a commutative ring. Thus, for example $(\omega^\omega + \omega^3 + \omega + 1) \oplus (\omega^3 + \omega) = \omega^\omega + \omega^3 2 + \omega 2 + 1$ and $(\omega^\omega + \omega^3 + \omega + 1) \otimes (\omega^3 + \omega) = \omega^{\omega+3} + \omega^{\omega+1} + \omega^6 + \omega^4 2 + \omega^3 + \omega^2 + \omega$.

More formally

Definition 2.1. *Let α and β be nonzero ordinals. With suitable re-labeling, the normal forms for these ordinals can be written using the same strictly decreasing set of exponents $\gamma_1 > \gamma_2 > \cdots > \gamma_n$. Thus $\alpha = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n$ and $\beta = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_n} n_n$ where $n_i, m_i \in \mathbb{Z}^+$. Then the* **natural sum** *and* **natural product** *of α and β are defined by*

$$\alpha \oplus \beta = \sum_i \omega^{\gamma_i}(m_i + n_i) \qquad \alpha \otimes \beta = \bigoplus_{ij} \omega^{\gamma_i \oplus \gamma_j} m_i n_j.$$

*In addition, we define $0 \oplus \alpha = \alpha \oplus 0 = \alpha$, and $0 \otimes \alpha = \alpha \otimes 0 = 0$ for any $\alpha \in$ **Ord**.*

The operations $\oplus$ and $\otimes$ are associative, commutative and cancellative: $(\alpha \oplus \gamma = \beta \oplus \gamma \implies \alpha = \beta)$ and $(\alpha \otimes \gamma = \beta \otimes \gamma \implies \alpha = \beta)$. The distributive law also holds: $\gamma \otimes (\alpha \oplus \beta) = (\gamma \otimes \alpha) \oplus (\gamma \otimes \beta)$. If α and β are finite ordinals then $\alpha \oplus \beta = \alpha + \beta$ and $\alpha \otimes \beta = \alpha\beta$ and these operations coincide with the usual addition and multiplication of natural numbers. For further information on the natural sum and product see [6, p. 107], [7], or [8, XIV - 28].

For our discussion of partially ordered sets we use the following notation: Let $\mathcal{L}$ be a partially ordered set and $x, y \in \mathcal{L}$. Then

$$\{\leq x\} = \{z \in \mathcal{L} \mid z \leq x\} \qquad \{\not\geq x\} = \{z \in \mathcal{L} \mid z \not\geq x\}$$
$$[x, y] = \{z \in \mathcal{L} \mid x \leq z \leq y\}$$

When it exists, the maximum (minimum) element of $\mathcal{L}$ will be labelled $\top$ ($\bot$).

A partially ordered set which satisfies the descending chain condition is called **well-founded** by many and **Artinian** by some. The latter name has the advantage of having an obvious dual for partially ordered sets which satisfy the ascending chain condition, namely **Noetherian**.

A totally ordered set which is Artinian is **well ordered**. The size of a well ordered set is very conveniently characterized by its order type which is an ordinal number. The next lemma is used to provide an analogous ordinal valued measure of the size of any Artinian partially ordered set.

Lemma 2.2. [5, Sect. 2][9, 3.2] *Let $\mathcal{L}$ be an Artinian partially ordered set. Then there is a unique function $\lambda_{\mathcal{L}}\colon \mathcal{L} \to \mathbf{Ord}$ satisfying the following equivalent conditions:*

1. *$\lambda_{\mathcal{L}}$ is strictly increasing, and, if $\lambda\colon \mathcal{L} \to \mathbf{Ord}$ is a strictly increasing function, then $\lambda_{\mathcal{L}}(x) \leq \lambda(x)$ for all $x \in \mathcal{L}$.*
2. *$\lambda_{\mathcal{L}}$ is strictly increasing and for any $x \in \mathcal{L}$ and $\alpha \in \mathbf{Ord}$ with $\alpha \leq \lambda_{\mathcal{L}}(x)$, there is some $y \leq x$ such that $\alpha = \lambda_{\mathcal{L}}(y)$.*
3. *For all $x \in \mathcal{L}$, $\lambda_{\mathcal{L}}(x) = \sup\{\lambda_{\mathcal{L}}(y) + 1 \mid y < x\}$.*

The ordinal $\lambda_{\mathcal{L}}(x)$ has various names in the literature: the rank, height [9],[10] or length [5] of x.

The image $\lambda_{\mathcal{L}}(\mathcal{L})$ is an initial segment of **Ord** so it is natural to define the rank, height or length of $\mathcal{L}$ to be the ordinal type of $\lambda_{\mathcal{L}}(\mathcal{L})$. This is done for example in [9]. In this paper we define the length of $\mathcal{L}$ only if $\mathcal{L}$ has a maximum element:

Definition 2.3. *An Artinian partially ordered set $\mathcal{L}$ is* **bounded** *if it has a maximum element $\top$. In this case, we define the* ***length*** *of $\mathcal{L}$ by* $\operatorname{len} \mathcal{L} = \lambda_{\mathcal{L}}(\top)$.

The following properties are basic:

Lemma 2.4. *Let $\mathcal{L}$ and $\mathcal{K}$ be bounded Artinian partially ordered sets.*

(1) *The order type of $\lambda_{\mathcal{L}}(\mathcal{L})$ is $\operatorname{len}\mathcal{L}+1$.*
(2) *If $\sigma\colon \mathcal{K}\to\mathcal{L}$ is strictly increasing, then $\operatorname{len}\mathcal{K}\leq\operatorname{len}\mathcal{L}$.*
(3) *If $x\in\mathcal{L}$, then $\operatorname{len}\{\leq x\}=\lambda_{\mathcal{L}}(x)$ and $\operatorname{len}\{\leq x\}+\operatorname{len}[x,\top]\leq\operatorname{len}\mathcal{L}$.*
(4) *$\mathcal{K}\times\mathcal{L}$ is a bounded Artinian partially ordered set, and for all (x,y) in $\mathcal{K}\times\mathcal{L}$, we have $\lambda_{\mathcal{K}\times\mathcal{L}}(x,y)=\lambda_{\mathcal{K}}(x)\oplus\lambda_{\mathcal{L}}(y)$. In particular,*

$$\operatorname{len}(\mathcal{K}\times\mathcal{L})=(\operatorname{len}\mathcal{K})\oplus(\operatorname{len}\mathcal{L}).$$

Proof. The proofs of 1-3 are easy [5, 2.6], and in any case, 2 follows easily from 2.2(1). For 4, see [9, 4-7.2], [5, 2.9] or [11]. □

If $\alpha\in\mathbf{Ord}$, then $\lambda_{\{\leq\alpha\}}$ is the identity map on $\{\leq\alpha\}$, and so, in particular, $\operatorname{len}\{\leq\alpha\}=\alpha$. This, together with 2.4(4), provides a useful characterization of the natural sum: If $\alpha,\beta\in\mathbf{Ord}$, then

$$\alpha\oplus\beta=\operatorname{len}(\{\leq\alpha\}\times\{\leq\beta\}).$$

For example, if $\gamma<\alpha\oplus\beta$, then from 2.2(2) there must be some $(\alpha',\beta')<(\alpha,\beta)$ such that $\gamma=\alpha'\oplus\beta'$. We will use this fact in proving Lemma 2.6.

A nonzero ordinal α is **decomposable** if there are ordinals $\beta_1,\beta_2<\alpha$ such that $\beta_1+\beta_2=\alpha$, otherwise α is **indecomposable**. Using normal forms it is easy to show the following:

Lemma 2.5. *For nonzero $\alpha\in\mathbf{Ord}$ the following are equivalent:*

(1) *α is indecomposable*
(2) $(\forall\beta_1,\beta_2\in\mathbf{Ord})\ (\beta_1,\beta_2<\alpha\implies\beta_1+\beta_2<\alpha)$
(3) $(\forall\beta_1,\beta_2\in\mathbf{Ord})\ (\beta_1,\beta_2<\alpha\implies\beta_1\oplus\beta_2<\alpha)$
(4) *$\alpha=\omega^{\gamma}$ for some $\gamma\in\mathbf{Ord}$*

Lemma 2.6. *If α and β are indecomposable ordinals, then*

$$\alpha\otimes\beta=\sup\{\alpha'\otimes\beta'\mid(\alpha',\beta')<(\alpha,\beta)\}.$$

Proof. Let $\alpha=\omega^{\delta}$ and $\beta=\omega^{\epsilon}$. Then $\alpha\otimes\beta=\omega^{\delta\oplus\epsilon}$ is a limit ordinal, so $\alpha\otimes\beta=\sup\{\mu\mid\mu<\alpha\otimes\beta\}$. To prove the claim it suffices to show that for any $\mu<\alpha\otimes\beta$, there is some $(\alpha',\beta')<(\alpha,\beta)$ such that $\mu\leq\alpha'\otimes\beta'$.

If the normal form for μ has leading term ω^{γ} and n terms, we have $\mu\leq\omega^{\gamma}n$ and also $\gamma<\delta\oplus\epsilon$. From above, $\gamma=\delta'\oplus\epsilon'$ for some $(\delta',\epsilon')<(\delta,\epsilon)$. We have two cases: If $\delta'<\delta$, then set $\alpha'=\omega^{\delta'}n<\omega^{\delta}=\alpha$ and $\beta'=\omega^{\epsilon'}\leq\omega^{\epsilon}=\beta$. If $\epsilon'<\epsilon$, then set $\beta'=\omega^{\epsilon'}n<\omega^{\epsilon}=\beta$ and $\alpha'=\omega^{\delta'}\leq\omega^{\delta}=\alpha$. In either case we have $(\alpha',\beta')<(\alpha,\beta)$ and $\mu\leq\omega^{\gamma}n=\alpha'\otimes\beta'$. □

We next consider the special case of modular lattices. A **lattice** is a partially ordered set $\mathcal{L}$ such that every pair of elements, $x, y \in \mathcal{L}$, has a supremum, $x \vee y$, and an infimum, $x \wedge y$. A lattice $\mathcal{L}$ is **modular** if

$$x_1 \leq x_2 \implies (x_1 \vee y) \wedge x_2 = x_1 \vee (y \wedge x_2)$$

for all $x_1, x_2, y \in \mathcal{L}$. Any Artinian lattice has a least element $\bot$, and, if bounded, has a maximum element $\top$. A bounded Artinian modular lattice $\mathcal{L}$ is **critical** if $\operatorname{len} \mathcal{L} = \omega^\gamma$ for some $\gamma \in \mathbf{Ord}$. A **critical series** for a bounded Artinian modular lattice $\mathcal{L}$, is a sequence $\bot = z_0 < z_1 < \cdots < z_n = \top$ in $\mathcal{L}$ such that $\operatorname{len}[z_{i-1}, z_i] = \omega^{\gamma_i}$ for all i, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$.

The most important property of a bounded Artinian modular lattice $\mathcal{L}$ is that if $x \in \mathcal{L}$ then $\operatorname{len} \mathcal{L}$ is controlled by $\operatorname{len}[\bot, x]$ and $\operatorname{len}[x, \top]$:

Lemma 2.7. [5, 3.2(1)] *Let x be an element of a bounded Artinian modular lattice $\mathcal{L}$. Then*

$$\operatorname{len}[\bot, x] + \operatorname{len}[x, \top] \leq \operatorname{len} \mathcal{L} \leq \operatorname{len}[\bot, x] \oplus \operatorname{len}[x, \top].$$

The first inequality is just 2.4(3). Consideration of when these inequalities are, in fact, equalities gives the following lemma:

Lemma 2.8. [5, 3.8] *Let $\mathcal{L}$ be a bounded Artinian modular lattice. Then the following are equivalent*

1. $\operatorname{len} \mathcal{L} = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ *in normal form.*
2. *$\mathcal{L}$ has a critical series $\bot = z_0 < z_1 < \cdots < z_n = \top$ with $\operatorname{len}[z_{i-1}, z_i] = \omega^{\gamma_i}$ for $i = 1, 2, \ldots, n$.*

Definition 2.9. *Let $\mathcal{L}$ be a partially ordered set. A* **lower set** *of $\mathcal{L}$ is a subset $D \subseteq \mathcal{L}$ such that*

$$x \leq y \in D \implies x \in D$$

for all $x, y \in \mathcal{L}$. We write $\Downarrow \mathcal{L}$ for the set of lower sets of $\mathcal{L}$ ordered by inclusion. $\Downarrow \mathcal{L}$ has minimum element $\bot = \emptyset$ and maximum element $\top = \mathcal{L}$. Since the union and intersection of a set of lower sets are also lower sets, $\Downarrow \mathcal{L}$ is a complete distributive lattice. In particular, $\Downarrow \mathcal{L}$ is a modular lattice [12, I and V].

Note that for any $x \in \mathcal{L}$, both $\{\leq x\}$ and $\{\not\geq x\}$ are lower sets of $\mathcal{L}$.

The following is standard. See for example [13, 2.21], [14], [15], [16, 1.4].

Lemma 2.10. *Let $\mathcal{L}$ be a partially ordered set. Then the following are equivalent:*

1. *$\Downarrow \mathcal{L}$ is Artinian.*
2. *For every infinite sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathcal{L}$ there are $i < j$ such that $a_i \leq a_j$.*
3. *$\mathcal{L}$ is Artinian and contains no infinite antichains.*

(4) *Every nonempty subset of $\mathcal{L}$ has a nonzero finite number of minimal elements.*
(5) *Every infinite subset of $\mathcal{L}$ contains an infinite strictly increasing sequence.*
(6) *Every infinite sequence in $\mathcal{L}$ contains an infinite increasing subsequence.*

A partially ordered set satisfying the conditions of this lemma is often called a **well partial ordering** in the literature. Notice that from 3 and 5 of this lemma, if $\mathcal{L}$ is Artinian, Noetherian and contains no infinite antichains, then $\mathcal{L}$ is finite.

Definition 2.11. *Let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered sets. We will write $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ for the set of decreasing functions from $\mathcal{K}$ to $\mathcal{L}$ ordered in the usual way: $\sigma_1 \leq \sigma_2$ if $\sigma_1(x) \leq \sigma_2(x)$ for all $x \in \mathcal{K}$.*

Of course, $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ is a partially ordered set. It is easy to see that $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ is bounded (a lattice, a modular lattice) if $\mathcal{L}$ is bounded (a lattice, a modular lattice).

Lemma 2.12. *Let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered sets.*

(1) $\Downarrow(\mathcal{K} \times \mathcal{L}) \cong \mathrm{dec}(\mathcal{K}, \Downarrow \mathcal{L}) \cong \mathrm{dec}(\mathcal{L}, \Downarrow \mathcal{K})$.
(2) *If $\Downarrow \mathcal{K}$ and $\Downarrow \mathcal{L}$ are Artinian, then $\Downarrow(\mathcal{K} \times \mathcal{L})$ is Artinian.*
(3) *If $\Downarrow \mathcal{K}$ and $\mathcal{L}$ are Artinian and $\sigma \in \mathrm{dec}(\mathcal{K}, \mathcal{L})$, then $\sigma(\mathcal{K})$ is finite.*
(4) *If $\Downarrow \mathcal{K}$ and $\mathcal{L}$ are Artinian, then $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ is Artinian.*

Proof.

(1) We define maps Φ: $\Downarrow(\mathcal{K} \times \mathcal{L}) \to \mathrm{dec}(\mathcal{K}, \Downarrow \mathcal{L})$ and Ψ: $\mathrm{dec}(\mathcal{K}, \Downarrow \mathcal{L}) \to \Downarrow(\mathcal{K} \times \mathcal{L})$ as follows: For $D \in \Downarrow(\mathcal{K} \times \mathcal{L})$, let
$$\Phi(D)(x) = \{y \in \mathcal{L} \mid (x, y) \in D\}$$
for $x \in \mathcal{K}$. For $\sigma \in \mathrm{dec}(\mathcal{K}, \Downarrow \mathcal{L})$, let
$$\Psi(\sigma) = \{(x, y) \in \mathcal{K} \times \mathcal{L} \mid y \in \sigma(x)\}.$$
It is routine to check that Φ and Ψ are inverse order isomorphisms.
(2) This easily follows from 2.10(6): Any infinite sequence in $\mathcal{K} \times \mathcal{L}$ has a subsequence in which the first entries are increasing. Then this subsequence has a subsequence in which the second entries are increasing too.
(3) Applying 2.10(2) to $\mathcal{K}$ we see that for any infinite sequence $(\sigma(a_n))_{n \in \mathbb{N}}$ in $\sigma(\mathcal{K})$ there are $i < j$ such that $\sigma(a_i) \geq \sigma(a_j)$. Thus from the dual version of 2.10(2 and 3), $\sigma(\mathcal{K})$ is Noetherian and contains no infinite antichains. Being a subset of $\mathcal{L}$, $\sigma(\mathcal{K})$ is also Artinian. As we noted above, this implies $\sigma(\mathcal{K})$ is finite.

(4) Let $\sigma_1 \geq \sigma_2 \geq \dots$ be a decreasing sequence of functions in $\operatorname{dec}(\mathcal{K}, \mathcal{L})$. Define $\sigma\colon \mathbb{N} \times \mathcal{K} \to \mathcal{L}$ by $\sigma(n, x) = \sigma_n(x)$. It is easy to confirm that σ is a decreasing function. Since $\Downarrow\mathbb{N}$ and $\Downarrow\mathcal{K}$ are Artinian, so is $\Downarrow(\mathbb{N} \times \mathcal{K})$ and hence, from 3, $Y = \sigma(\mathbb{N} \times \mathcal{K})$ is finite. For $y \in Y$, define $U_y = \sigma^{-1}(y)$. Then U_y is a nonempty subset of $\mathbb{N} \times \mathcal{K}$ and so has a finite set of minimal elements X_y. Set $X = \cup_{y \in Y} X_y$, a finite subset of $\mathbb{N} \times \mathcal{K}$.

Now choose N greater than any $n' \in \mathbb{N}$ such $(n', x') \in X$ for some $x' \in \mathcal{K}$. We will show that $\sigma_n = \sigma_N$ for all $n \geq N$. Let $x \in \mathcal{K}$ and $y = \sigma_n(x) \in Y$. Then $(n, x) \in U_y$, so there is some $(n', x') \in X_y$ such that $(n', x') \leq (n, x)$. In particular, $x' \leq x$. We now have $(n', x') \leq (N, x) \leq (n, x)$, and since σ is decreasing,

$$y = \sigma(n, x) \leq \sigma(N, x) \leq \sigma(n', x') = y.$$

Thus $\sigma_n(x) = \sigma_N(x)$.

□

It is worth noticing that the converses of 2 and 4 are almost always true: $\Downarrow\mathcal{K}$ embeds in $\Downarrow(\mathcal{K} \times \mathcal{L})$ via the map $D \mapsto D \times \mathcal{L}$. Therefore if $\Downarrow(\mathcal{K} \times \mathcal{L})$ is Artinian, then so are $\Downarrow\mathcal{K}$ and, by symmetry, $\Downarrow\mathcal{L}$. The lattice $\mathcal{L}$ embeds in $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ as the set of constant maps, so if $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ is Artinian, then $\mathcal{L}$ is too.

If $\mathcal{L}$ is an antichain, then $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ is an antichain (and hence is Artinian) for any $\mathcal{K}$ whatsoever. If $\mathcal{L}$ is not an antichain, containing elements $y_1 < y_2$ say, then $\Downarrow\mathcal{K}$ embeds in $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ via the map $D \mapsto \sigma_D$ where

$$\sigma_D(x) = \begin{cases} y_2 & \text{if } x \in D \\ y_1 & \text{if } x \notin D \end{cases}$$

for $x \in \mathcal{K}$. Thus if $\mathcal{L}$ is not an antichain and $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ is Artinian, then $\Downarrow\mathcal{K}$ is Artinian.

Theorem 2.13. *Let $\mathcal{K}$ be a partially ordered set such that $\Downarrow\mathcal{K}$ is Artinian, and $\mathcal{L}$ a bounded Artinian modular lattice. Then*

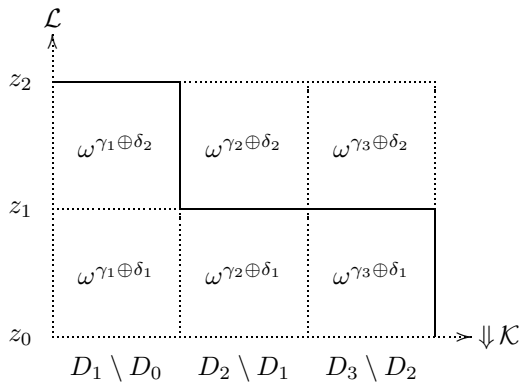
$$\operatorname{len}(\operatorname{dec}(\mathcal{K}, \mathcal{L})) = (\operatorname{len} \Downarrow\mathcal{K}) \otimes (\operatorname{len} \mathcal{L}).$$

Proof. Notice that $\Downarrow\mathcal{K}$, $\mathcal{L}$ and $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ are bounded Artinian modular lattices, so 2.8 applies to each. Proof is by induction on the pair of ordinals $(\operatorname{len} \Downarrow\mathcal{K}, \operatorname{len} \mathcal{L}) = (\alpha, \beta)$ in $\mathbf{Ord} \times \mathbf{Ord}$. The induction starts with the case $(\operatorname{len} \Downarrow\mathcal{K}, \operatorname{len} \mathcal{L}) = (1, 0)$ where $\mathcal{L}$, $\mathcal{K}$ and $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ contain one element each. Since $\operatorname{len} \operatorname{dec}(\mathcal{K}, \mathcal{L}) = 0$, the claim is trivially true.

For the induction step we have two cases:

- Suppose that at least one of the ordinals α or β is decomposable. Let $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_m}$ in normal form, and $\emptyset = D_0 < D_1 < \cdots < D_m = \mathcal{K}$ be a critical series for $\Downarrow \mathcal{K}$. Let $\beta = \omega^{\delta_1} + \omega^{\delta_2} + \cdots + \omega^{\delta_n}$ in normal form, and $\bot = z_0 < z_1 < \cdots < z_n = \top$ be a critical series for $\mathcal{L}$.

  Let $\mathbb{F}$ be the set of functions in $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ which are constant on the sets $D_1 \setminus D_0, D_2 \setminus D_1, \ldots, D_m \setminus D_{m-1}$ and whose image is in $\{z_0, z_1, \ldots, z_n\}$. These functions are best described using a diagram as below. For simplicity we have chosen $m = 3$ and $n = 2$.



  For example, the solid line is the graph of the function $\sigma \in \mathbb{F}$ defined by

$$\sigma(x) = \begin{cases} z_2 & \text{if } x \in D_1 \\ z_1 & \text{if } x \in D_3 \setminus D_1. \end{cases}$$

  The (i, j) entry in the diagram is $\omega^{\gamma_i \oplus \delta_j}$.

  We will prove that for any $\sigma \in \mathbb{F}$, $\mathrm{len}[\bot, \sigma]$ is the natural sum of all the entries in the diagram which lie below the graph of σ. In particular, since $\top \in \mathrm{dec}(\mathcal{K}, \mathcal{L})$ is in $\mathbb{F}$, this implies that $\mathrm{len}(\mathrm{dec}(\mathcal{K}, \mathcal{L}))$ is the natural sum of all the entries in the diagram, that is, $\mathrm{len}(\mathrm{dec}(\mathcal{K}, \mathcal{L})) = \alpha \otimes \beta$.

  The proof of this claim is by a further induction, this time on the number of squares below the graph of $\sigma \in \mathbb{F}$. If the number squares is zero then $\sigma = \bot \in \mathrm{dec}(\mathcal{K}, \mathcal{L})$ and so $\mathrm{len}[\bot, \sigma] = 0$.

  Now suppose the number of squares below the graph of σ is not zero. Pick one of these squares whose entry $\omega^{\gamma_i \oplus \delta_j}$ is minimum. Since the entries in the squares are decreasing from left to right

and from bottom to top, this square can, in addition, be chosen so that its right and top edges are on the graph of σ.

Let $\sigma' \in \mathbb{F}$ be the function whose graph is the same as that of σ except that the right and top edges of the chosen square are replaced by the left and bottom edges. Thus, σ' has one less square under its graph than σ and $\sigma' < \sigma$.

It is easy to see that $[\sigma', \sigma]$ is isomorphic to $\mathrm{dec}(D_i \setminus D_{i-1}, [z_{j-1}, z_j])$. Moreover, $\Downarrow(D_i \setminus D_{i-1}) \cong [D_{i-1}, D_i] \subseteq \Downarrow\mathcal{K}$, and hence

$$\operatorname{len} \Downarrow(D_i \setminus D_{i-1}) = \operatorname{len}[D_{i-1}, D_i] = \omega^{\gamma_i}.$$

Since $(\omega^{\gamma_i}, \omega^{\delta_j}) < (\alpha, \beta)$ we have

$$\begin{aligned} \operatorname{len}[\sigma', \sigma] &= \operatorname{len} \mathrm{dec}(D_i \setminus D_{i-1}, [z_{j-1}, z_j]) \\ &= (\operatorname{len} \Downarrow(D_i \setminus D_{i-1}) \otimes (\operatorname{len}[z_{j-1}, z_j]) \\ &= \omega^{\gamma_i} \otimes \omega^{\delta_j} = \omega^{\gamma_i \oplus \delta_j}, \end{aligned}$$

which is the entry in the diagram representing the difference of the two graphs.

By induction, $\operatorname{len}[\bot, \sigma']$ is the natural sum of all the entries in the diagram which lie below the graph of σ'. Since $\omega^{\gamma_i \oplus \delta_j}$ is less than or equal to these entries, we have $\operatorname{len}[\bot, \sigma'] + \omega^{\gamma_i \oplus \delta_j} = \operatorname{len}[\bot, \sigma'] \oplus \omega^{\gamma_i \oplus \delta_j}$, and hence $\operatorname{len}[\bot, \sigma'] + \operatorname{len}[\sigma', \sigma] = \operatorname{len}[\bot, \sigma'] \oplus \operatorname{len}[\sigma', \sigma]$. But, applying 2.7 to the lattice $[\bot, \sigma]$, we also have

$$\operatorname{len}[\bot, \sigma'] + \operatorname{len}[\sigma', \sigma] \leq \operatorname{len}[\bot, \sigma] \leq \operatorname{len}[\bot, \sigma'] \oplus \operatorname{len}[\sigma', \sigma].$$

Hence these inequalities are equalities and $\operatorname{len}[\bot, \sigma] = \operatorname{len}[\bot, \sigma'] \oplus \operatorname{len}[\sigma', \sigma]$ is the natural sum of all the entries in the diagram which lie below the graph of σ.

- Suppose that both α and β are indecomposable. We will show first that $\alpha \otimes \beta \leq \operatorname{len}(\mathrm{dec}(\mathcal{K}, \mathcal{L}))$. From 2.6 it suffices to show $\alpha' \otimes \beta' \leq \operatorname{len}(\mathrm{dec}(\mathcal{K}, \mathcal{L}))$ for all $(\alpha', \beta') < (\alpha, \beta)$:

  From 2.2(2) there are $D \in \Downarrow\mathcal{K}$ such that $\lambda_{\Downarrow\mathcal{K}}(D) = \alpha'$, and $x \in \mathcal{L}$ such that $\lambda_{\mathcal{L}}(x) = \beta'$. By induction and 2.4(3) we have $\operatorname{len}(\mathrm{dec}(D, [\bot, x]) = \alpha' \otimes \beta'$. Since $\mathrm{dec}(D, [\bot, x])$ is contained in $\mathrm{dec}(\mathcal{K}, \mathcal{L})$ we have $\alpha' \otimes \beta' \leq \operatorname{len}(\mathrm{dec}(\mathcal{K}, \mathcal{L}))$ as required.

  For the opposite inequality, it suffices to show that for any $\sigma \in \mathrm{dec}(\mathcal{K}, \mathcal{L})$ with $\sigma < \top$, we have $\operatorname{len}[\bot, \sigma] < \alpha \otimes \beta$.

  Since $\sigma < \top$, there is some $x \in \mathcal{K}$ such that $\sigma(x) \neq \top$ in $\mathcal{L}$. Set $\beta' = \lambda_{\mathcal{L}}(\sigma(x)) < \beta$ and $D = \{\not\geq x\} \in \Downarrow\mathcal{K}$. Since $x \notin D$ we have

$\alpha' = \lambda_{\Downarrow\mathcal{K}}(D) < \alpha$. Define $\sigma' \in \operatorname{dec}(\mathcal{K}, \mathcal{L})$ by

$$\sigma'(y) = \begin{cases} \top & \text{if } y \in D \\ \sigma(x) & \text{if } y \notin D \end{cases}$$

and notice that $\sigma \leq \sigma' < \top$. We calculate a bound for $\operatorname{len}[\bot, \sigma']$ using the sequence $\bot < \sigma'' < \sigma'$ where σ'' is the constant function on $\mathcal{K}$ with image $\sigma(x)$. We have $[\bot, \sigma''] = \operatorname{dec}(\mathcal{K}, [\bot, \sigma(x)])$ and so $\operatorname{len}[\bot, \sigma''] = \alpha \otimes \beta'$. The interval $[\sigma'', \sigma']$ is easily seen to be isomorphic with $\operatorname{dec}(D, [\sigma(x), \top])$. Hence $\operatorname{len}[\sigma'', \sigma'] = \beta' \otimes (\operatorname{len}[\sigma(x), \top]) \leq \beta' \otimes \alpha$. Applying 2.7 to the lattice $[\bot, \sigma']$ we get $\operatorname{len}[\bot, \sigma'] \leq \operatorname{len}[\bot, \sigma''] \oplus \operatorname{len}[\sigma'', \sigma']$. Thus finally,

$$\begin{aligned} \operatorname{len}[\bot, \sigma] \leq \operatorname{len}[\bot, \sigma'] &\leq \operatorname{len}[\bot, \sigma''] \oplus \operatorname{len}[\sigma'', \sigma'] \\ &\leq (\alpha \otimes \beta') \oplus (\beta' \otimes \alpha) < \alpha \otimes \beta \end{aligned}$$

Since $\alpha \otimes \beta$ is indecomposable and $\alpha \otimes \beta', \beta' \otimes \alpha < \alpha \otimes \beta$, the last inequality follows from 2.5.

□

Corollary 2.14. *Let $\mathcal{K}$ be a partially ordered set such that $\Downarrow\mathcal{K}$ is Artinian, and $\mathcal{L}$ a bounded Artinian partially ordered set. Then*

$$\operatorname{len}(\operatorname{dec}(\mathcal{K}, \mathcal{L})) \leq (\operatorname{len} \Downarrow \mathcal{K}) \otimes (\operatorname{len} \mathcal{L})$$

with equality in the following situations:

1. *$\mathcal{L}$ is a modular lattice.*
2. *$\mathcal{L}$ contains a chain of length $\operatorname{len}\mathcal{L}$.*
3. *$\Downarrow\mathcal{L}$ is Artinian.*
4. *$\operatorname{len}\mathcal{L}$ is finite.*

Proof. It is easy to see that the strictly increasing map $\lambda_{\mathcal{L}} \colon \mathcal{L} \to \{\leq \operatorname{len}\mathcal{L}\}$ induces a strictly increasing map from $\operatorname{dec}(\mathcal{K}, \mathcal{L})$ to $\operatorname{dec}(\mathcal{K}, \{\leq \operatorname{len}\mathcal{L}\})$. Since $\{\leq \operatorname{len}\mathcal{L}\}$ is trivially a modular lattice, we have from 2.4(2) and 2.13 that $\operatorname{len}(\operatorname{dec}(\mathcal{K}, \mathcal{L})) \leq \operatorname{len}(\operatorname{dec}(\mathcal{K}, \{\leq \operatorname{len}\mathcal{L}\}) = (\operatorname{len} \Downarrow \mathcal{K}) \otimes (\operatorname{len}\mathcal{L})$.

1. From 2.13.
2. A strictly increasing map from $\{\leq \operatorname{len}\mathcal{L}\}$ to $\mathcal{L}$ induces a strictly increasing map from $\operatorname{dec}(\mathcal{K}, \{\leq \operatorname{len}\mathcal{L}\})$ to $\operatorname{dec}(\mathcal{K}, \mathcal{L})$. Thus, from 2.4(2) and 2.13, $\operatorname{len}(\operatorname{dec}(\mathcal{K}, \mathcal{L})) \geq \operatorname{len}(\operatorname{dec}(\mathcal{K}, \{\leq \operatorname{len}\mathcal{L}\}) = (\operatorname{len} \Downarrow \mathcal{K}) \otimes (\operatorname{len}\mathcal{L})$.
3. For any ordinal $\alpha \in \lambda_{\mathcal{L}}(\mathcal{L})$, the inverse image $\lambda_{\mathcal{L}}^{-1}(\alpha)$ is an antichain, so by 2.10(3) is finite. It then follows from [9, 4-4.5] that $\mathcal{L}$ contains a chain of length $\operatorname{len}\mathcal{L}$. The claim then follows from 2.

(4) If $\operatorname{len}\mathcal{L}$ is finite, then $\mathcal{L}$ contains a chain of length $\operatorname{len}\mathcal{L}$, and the claim follows from 2.

□

Corollary 2.15. *Let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered sets such that $\Downarrow\mathcal{K}$ and $\Downarrow\mathcal{L}$ are Artinian. Then*

$$\operatorname{len}(\Downarrow(\mathcal{K}\times\mathcal{L})) = \operatorname{len}(\Downarrow\mathcal{K})\otimes\operatorname{len}(\Downarrow\mathcal{L}).$$

Proof. This follows directly from 2.12(1) and 2.13. □

This corollary is also a simple consequence (as pointed out in [9] and [10, 5.4]) of a theorem of De Jongh and Parikh [17, 3.5].

It is easy to check that for $\alpha\in\mathbf{Ord}$, we have $\Downarrow\{<\alpha\}\cong\{\leq\alpha\}$ and hence $\operatorname{len}(\Downarrow\{<\alpha\})=\alpha$. Thus, from 2.15, we have another characterization of the natural product of ordinals: If $\alpha,\beta\in\mathbf{Ord}$, then

$$\alpha\otimes\beta = \operatorname{len}\Downarrow(\{<\alpha\}\times\{<\beta\}).$$

This parallels the characterization of the natural sum:

$$\alpha\oplus\beta = \operatorname{len}(\{\leq\alpha\}\times\{\leq\beta\}).$$

3. Noetherian Modules and Polynomial Rings

Let A be a Noetherian left module over a ring R and $\mathcal{L}(A)$ the lattice of submodules of A (ordered by inclusion). Then the dual of this lattice, $\mathcal{L}^\circ(A)$, is a bounded modular Artinian lattice. In particular, using 2.3 we can define the **length** of A by $\operatorname{len}A=\operatorname{len}(\mathcal{L}^\circ(A))$. For a discussion of the basic properties of this measure of the size of Noetherian modules see [1] and [5].

If R is a left Noetherian ring, then so is $R[x]$, the polynomial ring over R. Thinking of R and $R[x]$ as left modules over themselves we define $\operatorname{len}R$ and $\operatorname{len}R[x]$ as above.

The main theorem of this paper, which follows, relates these two lengths. Its proof is a version of the usual proof of the Hilbert Basis Theorem:

Theorem 3.1. *If R is a left Noetherian ring, then*

$$\operatorname{len}R[x] = \omega\otimes\operatorname{len}R.$$

Proof. We define a map $\Phi\colon\mathcal{L}^\circ(R[x])\to\operatorname{dec}(\mathbb{Z}^+,\mathcal{L}^\circ(R))$ as follows: For a left ideal I of $R[x]$, let

$$\Phi(I)(n) = \{a_n \mid a_0+a_1x+\ldots+a_nx^n\in I\}$$

for $n\in\mathbb{Z}^+$. It is routine to check that $\Phi(I)(n)$ is a left ideal of R, and that $\Phi(I)(0)\leq\Phi(I)(1)\leq\Phi(I)(2)\ldots$, that is, $\Phi(I)\in\operatorname{dec}(\mathbb{Z}^+,\mathcal{L}^\circ(R))$.

We now show that Φ is a strictly increasing function: Since it is clear that Φ is increasing it suffices to show that if $J \leq I \leq R[x]$ are such that $\Phi(I) = \Phi(J)$, then $I = J$.

Suppose not. Let $f = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial of smallest degree in $I \setminus J$. Since $a_n \in \Phi(I)(n) = \Phi(J)(n)$, there is some polynomial $g \in J \leq I$ having a_nx^n as its leading term. Since both f and g are in I, so is $f - g$. The degree of $f - g$ is less than n, so by the minimality of n, $f - g \in J$. But this implies $f \in J$, which is contrary to assumption.

Applying 2.4(2) to the map Φ and using 2.13 we get

$$\begin{aligned}\operatorname{len} R[x] = \operatorname{len} \mathcal{L}^\circ(R[x]) &\leq \operatorname{len}(\operatorname{dec}(\mathbb{Z}^+, \mathcal{L}^\circ(R))) \\ &= (\operatorname{len} \Downarrow \mathbb{Z}^+) \otimes (\operatorname{len} \mathcal{L}^\circ(R)) = \omega \otimes \operatorname{len} R.\end{aligned}$$

To prove the opposite inequality we define a map $\Psi\colon \operatorname{dec}(\mathbb{Z}^+, \mathcal{L}^\circ(R)) \to \mathcal{L}^\circ(R[x])$. For $\sigma \in \operatorname{dec}(\mathbb{Z}^+, \mathcal{L}^\circ(R))$, let

$$\Psi(\sigma) = \{a_0 + a_1x + \ldots + a_nx^n \mid n \in \mathbb{Z}^+ \text{ and } a_i \in \sigma(i) \text{ for all } i \leq n\}.$$

It is routine to check that, since σ is decreasing, $\Psi(\sigma) \in \mathcal{L}^\circ(R[x])$. Since Ψ is strictly increasing, we can apply 2.4(2) and 2.13 as above to get $\operatorname{len} R[x] \geq \omega \otimes (\operatorname{len} R)$. □

As mentioned in the introduction, the Krull dimensions of R and $R[x]$ are encoded in their lengths as the degree of the normal forms. With a small amount of ordinal arithmetic, the reader can easily show that $\operatorname{Kdim} R[x] = \operatorname{Kdim} R + 1$ follows from 3.1.

Of course, we have $R[x_1, x_2, \ldots, x_n] \cong R[x_1][x_2]\ldots[x_n]$ and so

Corollary 3.2. *If R is a left Noetherian ring, then*

$$\operatorname{len} R[x_1, x_2, \ldots, x_n] = \omega^n \otimes \operatorname{len} R.$$

4. Conjecture and Speculation

The polynomial rings $R[x]$ and $R[x_1, x_2, \ldots, x_n]$ are examples of monoid rings. These are constructed as follows:

Let R be a ring, and M a commutative monoid, whose operation we will write additively. Then the monoid ring $R[M]$ is the set of functions from M to R which have finite support, that is $f\colon M \to R$ is in $R[M]$ if $f(m) = 0$ for all but a finite number of elements $m \in M$. Addition and multiplication of elements of $R[M]$ are defined by $(f+g)(m) = f(m)+g(m)$ and $(fg)(m) = \sum_{m_1+m_2=m} f(m_1)g(m_2)$ for $f, g \in R[M]$. It is routine to check that $R[M]$ with these operations is a ring.

If $M = \mathbb{Z}^+$ with addition as monoid operation, then $R[M] \cong R[x]$, and if $M = (\mathbb{Z}^+)^n = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \ldots \times \mathbb{Z}^+$ (n factors), then $R[M] \cong R[x_1, x_2, \ldots, x_n]$.

In view of 3.1 and 3.2, it is natural to speculate that there ought to be a class of monoids such that, if R is left Noetherian, then so is $R[M]$ and

$$\operatorname{len} R[M] = l(M) \otimes (\operatorname{len} R) \tag{1}$$

where $l(M)$ is some ordinal valued measure of the size of M. Evidently, $l(\mathbb{Z}^+) = \omega$ and $l((\mathbb{Z}^+)^n) = \omega^n$. In general, for any monoid M of this type, $l(M) = \operatorname{len} k[M]$ where k is any field.

Is $l(M)$ the length of some Artinian partially ordered set? Certainly any commutative monoid M has a preorder defined by $a \leq b$ if $a+c = b$ for some $c \in M$. This is called the **algebraic preorder** or the **minimal preorder** on M. Both $\mathbb{Z}^+$ and $(\mathbb{Z}^+)^n$ are in fact partially ordered by $\leq$, and it seems that the same must be true of any M satisfying (1). For example, let M be the cyclic group with three elements. Then $a \leq b \leq a$ for any $a, b \in M$ so $\leq$ is not a partial order on M. For the field of complex numbers $\mathbb{C}$ we have $\mathbb{C}[M] \cong \mathbb{C}^3$ and hence $\operatorname{len} \mathbb{C}[M] = 3$, and for the field of real numbers $\mathbb{R}$ we have $\mathbb{R}[M] \cong \mathbb{C} \times \mathbb{R}$ and hence $\operatorname{len} \mathbb{R}[M] = 2$. Thus M cannot satisfy (1).

One half of the proof of 3.1 generalizes easily to monoid rings if $(M, \leq)$ is partially ordered:

Lemma 4.1. *Let R be a nontrivial ring and M a commutative monoid such that $\leq$ is a partial order. If $R[M]$ is left Noetherian, then R is left Noetherian, $\Downarrow M$ is Artinian and*

$$\operatorname{len} R[M] \geq (\operatorname{len} \Downarrow M) \otimes (\operatorname{len} R).$$

Proof. The ring R is the image of $R[M]$ under the ring homomorphism $f \mapsto \sum_{m \in M} f(m)$. Hence, if $R[M]$ is left Noetherian, then so is R.

For a lower set $D \subseteq M$, let $I_D = \{f \in R[M] \mid f(D) = \{0\}\}$. It is easy to check that I_D is a left ideal of $R[M]$ and that the map $D \mapsto I_D$ is an embedding of $\Downarrow M$ in $\mathcal{L}^\circ(R[M])$. Since $\mathcal{L}^\circ(R[M])$ is Artinian, so is $\Downarrow M$.

Following the pattern in 3.1, we define Ψ: $\operatorname{dec}(M, \mathcal{L}^\circ(R)) \to \mathcal{L}^\circ(R[M])$ by

$$\Psi(\sigma) = \{f \in R[M] \mid f(m) \in \sigma(m) \text{ for all } m \in M\}$$

for $\sigma \in \operatorname{dec}(M, \mathcal{L}^\circ(R))$. As in 3.1, Ψ is strictly increasing, so from 2.4(2) and 2.13 we get

$$\operatorname{len} R[M] \geq (\operatorname{len} \Downarrow M) \otimes (\operatorname{len} R).$$

□

This suggests that $l(M)$ from (1) might be the length of the Artinian partially ordered set $\Downarrow M$. Indeed $\operatorname{len}(\Downarrow \mathbb{Z}^+) = \omega$ and, from 2.15, $\operatorname{len}(\Downarrow(\mathbb{Z}^+)^n) = \omega^n$, so we have

$$\operatorname{len} R[M] = (\operatorname{len} \Downarrow M) \otimes (\operatorname{len} R) \tag{2}$$

when M is $\mathbb{Z}^+$ or $(\mathbb{Z}^+)^n$.

The other half of the proof of 3.1 makes use of the fact that $\mathbb{Z}^+$ is cancellative and well ordered. It follows then from this proof that (2) is true of any cancellative commutative monoid M such that $(M, \leq)$ is well ordered. Unfortunately an easy exercise shows that the only such monoids are $\mathbb{Z}^+$ and the trivial monoid $\{0\}$.

Of course the example $M = (\mathbb{Z}^+)^n$ shows that (2) may be true for monoids which are not totally ordered. There are also examples of noncancellative monoids for which (2) is true:

Example 4.2. *Let M be a finite monoid such that $\leq$ is a total order, $M = \{m_1, m_2, m_3, \ldots, m_n\}$ say, in increasing order. Then $m_1 = 0$ and $m_i + m_n = m_n$ for all i, so M is cancellative only if it is trivial. We also have $\operatorname{len} \Downarrow M = |M| = n$.*

For a left ideal $I \in \mathcal{L}^\circ(R[M])$ define a map $\Phi(I)$ from M to $\mathcal{L}^\circ(R)$ by

$$\Phi(I)(m) = \{f(m) \mid f \in I \text{ and } f(m') = 0 \text{ for all } m' > m\}$$

for $m \in M$. We have no reason to expect that $\Phi(I)$ is a decreasing map, but we can certainly consider Φ as a map from $\mathcal{L}^\circ(R[M])$ to $(\mathcal{L}^\circ(R))^n$. Moreover, the same argument used in 3.1 shows that Φ is strictly increasing. From 2.4(2, 4) and 2.13, we then get

$$\begin{aligned} \operatorname{len} R[M] &= \operatorname{len} \mathcal{L}^\circ(R[M]) \\ &\leq \operatorname{len}(\mathcal{L}^\circ(R))^n \\ &= \operatorname{len} \mathcal{L}^\circ(R) \oplus \operatorname{len} \mathcal{L}^\circ(R) \oplus \ldots \oplus \operatorname{len} \mathcal{L}^\circ(R) \ (n \text{ times}) \\ &= (\operatorname{len} \Downarrow M) \otimes \operatorname{len} R \end{aligned}$$

The opposite inequality is directly from 4.1.

This example also suggests that $\operatorname{len} R[M]$ depends only on the order induced by the monoid operation, and that other details of the monoid operation are irrelevant.

The above discussion leads one to conjecture that (2) is true for all monoids M such that $\leq$ is a partial order. More precisely,

Conjecture 4.3. *Let R be a nontrivial ring and M a commutative monoid such that $\leq$ is a partial order. Then $R[M]$ is left Noetherian if and only if*

R is left Noetherian and $\Downarrow M$ is Artinian. Moreover, in this circumstance,

$$\operatorname{len} R[M] = (\operatorname{len} \Downarrow M) \otimes (\operatorname{len} R).$$

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