WHEN GRADED DOMAINS ARE SCHREIER OR PRE-SCHREIER

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ABSTRACT. We characterize, in terms of properties of homogeneous elements, when a graded domain is pre-Schreier or Schreier. As a consequence, the following properties of a commutative monoid domain $A[M]$ are equivalent: (1) $A[M]$ is pre-Schreier; (2) $A[M]$ is Schreier; (3) A and M are Schreier. This is in contrast to pre-Schreier monoids and pre-Schreier integral domains, which need not be Schreier.

All rings in this paper will be commutative. An often studied problem in ring theory is to determine to what extent properties of a graded ring are determined by properties of its homogeneous elements. For example, see [1, 2, 3, 13, 14, 15, 19].

In this paper we investigate when an integral domain $R = \oplus_{m\in M} R_m$, graded by a cancellative torsion-free commutative monoid M, is pre-Schreier or Schreier, and then specialize to the case that R is a monoid domain $A[M]$.

To describe our results, we first recall the needed definitions.

A cancellative torsion-free monoid M is **integrally closed** if $nx \in M$ for $n \in \mathbb{N}$ and $x \in G$, the quotient group of M, implies $x \in M$. This condition can be reformulated in terms of the **natural preorder** $\leq$ on M which is defined by $x \leq y$ if $x + z = y$ for some $z \in M$. Thus M is integrally closed if and only if $nx \leq ny$ for $x, y \in M$ and $n \in \mathbb{N}$ implies $x \leq y$. By [11, 12.11], a monoid domain $A[M]$ is integrally closed if and only if A and M are integrally closed.

An element x of an integral domain R is **primal** if whenever x divides y_1y_2, with $y_1, y_2 \in R$, then $x = z_1z_2$ where z_1 divides y_1 and z_2 divides y_2. An integral domain in which each element is primal is **pre-Schreier**, and an integrally closed pre-Schreier domain is **Schreier**. Such rings play a role in various questions in ring theory. For example, see [5, 6, 7, 8, 18, 20, 21].

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There are also conditions corresponding to pre-Schreier and Schreier for monoids:

A monoid M has **decomposition** or is a **decomposition monoid**, if, for all $x, y_1, y_2 \in M$ such that $x \leq y_1 + y_2$, there are $z_1, z_2 \in M$ such that $x = z_1 + z_2$ with $z_1 \leq y_1$ and $z_2 \leq y_2$. Notice that a domain R is pre-Schreier if and only if the multiplicative monoid $R^* = R \setminus \{0\}$ has decomposition. Thus we will say that a cancellative torsion-free monoid is **pre-Schreier** if it has decomposition, and is **Schreier** if it is pre-Schreier and integrally closed. Such monoids arise in many contexts, often under different names. For example, see [4], [10] and [12].

Our main results characterize both pre-Schreier and Schreier M-graded domains in terms of homogeneous elements. In the case that R is a monoid domain $A[M]$, we get that $A[M]$ is pre-Schreier if and only if it is Schreier, and this holds if and only if A and M are Schreier. This is perhaps unexpected since pre-Schreier domains need not be Schreier ([6, pp. 11-12], [18, p. 80] or [20]) and pre-Schreier monoids need not be Schreier ([12, Example 2.23][1]).

In the next section a condition on a monoid M, which was used by Matsuda [17] in a characterization of when a monoid domain $A[M]$ is Schreier, is shown to be equivalent to M being pre-Schreier. In Section 2 the above mentioned characterizations of M-graded pre-Schreier and Schreier domains are given, and in Section 3 we give the above characterization of Schreier and pre-Schreier monoid domains.

1. Cancellative Decomposition Monoids

Matsuda [17] has shown that a monoid ring $A[M]$ is Schreier if and only if A is Schreier, M is integrally closed, and M satisfies the following condition:

> $(*)$ For any nonempty finite subsets $Y_1, Y_2 \subseteq M$ and $x \in M$ such that $x \leq Y_1 + Y_2$, there are $z_1, z_2 \in M$ such that $x = z_1 + z_2$ with $z_1 \leq Y_1$ and $z_2 \leq Y_2$.

Here, if X and Y are subsets of a monoid M, then

$$X + Y = \{x + y \mid x \in X \text{ and } y \in Y\},$$

[1]This example was brought to our attention by K. R. Goodearl, after seeing our more complicated example.

and if $x \in M$, then $x \leq Y$ if $x \leq y$ for all $y \in Y$. Thus $x \leq Y$ if and only if there is a subset $Z \subseteq M$ such that $Y = x + Z$. If M is cancellative, it follows that $|Y| = |Z|$ in the above.

Clearly, for any monoid M, condition $(*)$ implies M is a decomposition monoid. In the next lemma we see that the converse is true for cancellative monoids.

Lemma 1.1. *If M is a cancellative decomposition monoid, then M satisfies $(*)$.*

Proof. The proof is by induction on $n = |Y_1| + |Y_2|$. The case $n = 2$, when $|Y_1| = |Y_2| = 1$, is true by definition of decomposition.

Suppose the claim is true for all $n \leq N$ for some $N \in \mathbb{N}$. To prove the claim for $n = N + 1$, we will suppose that there are nonempty subsets $Y_1, Y_2 \subseteq M$ with $N = |Y_1|+|Y_2|$ and $x, y_1 \in M$ such that $x \leq (Y_1 \cup \{y_1\})+Y_2$, and show that $x = z_1 + z_2$ for some $z_1, z_2 \in M$ such that $z_1 \leq Y_1 \cup \{y_1\}$ and $z_2 \leq Y_2$.

By hypothesis we have $x \leq Y_1 + Y_2$ and $x \leq y_1 + Y_2$. By the induction hypothesis, the first of these inequalities implies that there are $w_1, w_2 \in M$ such that $x = w_1+w_2$ with $w_1 \leq Y_1$ and $w_2 \leq Y_2$. In particular, $Y_2 = w_2+Z_2$ for some subset $Z_2 \subseteq M$ with $|Z_2| = |Y_2|$. From the second inequality we now get $w_1 + w_2 = x \leq y_1 + Y_2 = y_1 + w_2 + Z_2$. Since the monoid is cancellative, this implies $w_1 \leq y_1 + Z_2$. We have $|\{y_1\}| + |Z_2| = 1 + |Y_2| \leq N$, so by the induction hypothesis there are z_1 and w_3 such that $w_1 = z_1 + w_3$ with $z_1 \leq y_1$ and $w_3 \leq Z_2$. Set $z_2 = w_2 + w_3$.

Now $x = w_1 + w_2 = z_1 + w_3 + w_2 = z_1 + z_2$. We have $z_1 \leq y_1$ and also $z_1 \leq w_1 \leq Y_1$, that is, $z_1 \leq Y_1 \cup \{y_1\}$. Finally, $z_2 = w_2 + w_3 \leq w_2 + Z_2 = Y_2$, as required. □

Combining this lemma with Matsuda's result, we have that a monoid ring $A[M]$ is Schreier if and only if A and M are Schreier. This result will appear in Theorem 3.2 as a consequence of our theorems about graded domains.

2. Pre-Schreier and Schreier graded domains

Let M be a cancellative torsion-free commutative monoid with quotient group G, $R = \oplus_{m \in M} R_m$, an M-graded domain, and S, the set of nonzero

homogeneous elements of R. Then the ring of fractions R_S is G-graded where, if $g \in G$, the homogeneous component R_g is $\{r/s \mid r \in R_m, s \in R_n, m, n \in M \text{ and } m - n = g\}$. We denote by (r) the principal ideal of R generated by r, and if A, B are ideals of R, we let $A :_R B = \{r \in R \mid rB \subseteq A\}$. In this section our characterizations of when R is pre-Schreier or Schreier are given. These characterizations use the property

> $(**)$ $I = (s) :_R (x)$ is a homogeneous ideal for each $s \in S$ and $x \in R$.

This property is closely related to R being integrally closed. In [2] an M-graded domain R as above is said to be **almost normal** if R contains each homogeneous element of R_S of nonzero degree which is integral over R. It is shown in [2, Theorems 3.2 and 3.5] that

> (AA) R integrally closed $\Rightarrow (**) \Rightarrow R$ almost normal.

Neither of these implications is reversible for domains graded by a cancellative torsion-free monoid M, as is shown in [2, p. 556] for the first implication, and in [16, Theorem 6] for the second implication. However, by [2, Theorem 3.2 and the proof of Theorem 3.9], these conditions are equivalent if R is a monoid domain $A[M]$ (which is M-graded by letting the homogeneous elements of degree m for $m \in M$ be the elements of the form rX^m, $r \in A$).

We say that a homogeneous element x, of an M-graded integral domain R, is **gr-primal** if whenever x divides y_1y_2, with $y_1, y_2 \in R$ homogeneous, then $x = z_1z_2$ where z_1 divides y_1 and z_2 divides y_2. If each homogeneous element is gr-primal we say R is **gr-pre-Schreier**. That is, R is gr-pre-Schreier if and only if S, as a multiplicative monoid, has decomposition.

For M-graded domains, we have the following.

Theorem 2.1. *Let M be a cancellative torsion-free commutative monoid, and $R = \oplus_{m \in M} R_m$ an M-graded domain. Then the following are equivalent:*

1. *R is pre-Schreier.*
2. *The homogeneous elements of R are primal.*
3. *R is gr-pre-Schreier and satisfies $(**)$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ R is gr-pre-Schreier are clear. Let S be the set of nonzero homogeneous elements of R.

(2) $\Rightarrow$ (1): By [1, Proposition 3.3], R_S is a GCD-domain, and thus is Schreier. By the proof of [5, Theorem 2.6(ii)], R is pre-Schreier.

(2) $\Rightarrow$ (3): It remains to show that R satisfies $(**)$. Suppose that $y \in I = (s) :_R (x)$ for some $s \in S$ and $x \in R$. Then $yx = zs$ for some $z \in R$. Since s is primal we can write $s = s_1 s_2$ with $y = s_1 y'$ and $x = s_2 x'$. Then y is contained in the homogeneous ideal (s_1). Moreover, since $s_1 x = s_1 s_2 x' = sx' \in (s)$, we also have $(s_1) \subseteq I$. It follows that I is homogeneous.

(3) $\Rightarrow$ (2): For $x \in R = \oplus_{m \in M} R_m$ with $x = x_1 + \cdots + x_n$, $x_i \in R_{m_i}$, $m_i \in M$ and $m_i \neq m_j$ for $i \neq j$, let $C(x)$ denote the homogeneous ideal $(x_1, \ldots x_n)$. Now suppose $s \in S$ divides xy with x and y in R. Then s divides every element of $C(xy)$, and, since R satisfies $(**)$, s divides every element of $C(x)C(y)$ [2, Theorem 3.2]. That is, if X and Y are the sets of non-zero homogeneous components of x and y respectively, then, in the notation of Section 1 applied to the multiplicative monoid S, we have $s \leq XY = \{z_1 z_2 \mid z_1 \in X \text{ and } z_2 \in Y\}$. Since R is a gr-pre-Schreier domain, S is a cancellative decomposition monoid, and so, by Lemma 1.1, $s = x'y'$ for some $x', y' \in S$ such that $x' \leq X$ and $y' \leq Y$. Since these equations imply that x' divides x and y' divides y, we have shown that s is primal. $\square$

In particular, this theorem says that any pre-Schreier M-graded domain satisfies $(**)$. In view of the connection between $(**)$ and being integrally closed, one wonders whether all such rings are integrally closed, and hence are Schreier. The following example shows that this is not always true.

Let A be a pre-Schreier domain that is not Schreier ([6, pp. 11-12], [18, p. 80] or [20]), let K be the quotient field of A and let X be an indeterminate. By [9, Theorem 2.7], the $\mathbb{Z}^+$-graded domain $R = A + XK[X]$ is pre-Schreier and R satisfies $(**)$ by [2, Theorem 3.2 and bottom of page 556]. But by [9, Corollary 2.9], R is not Schreier.

Let R again be an M-graded domain. If R is almost normal and the component R_0 is integrally closed in $(R_S)_0$, then R is integrally closed. Thus, the following characterization of the Schreier property is an immediate consequence of (AA) and Theorem 2.1.

Theorem 2.2. *Let M be a cancellative torsion-free commutative monoid, $R = \oplus_{m \in M} R_m$ be an M-graded domain and S denote the set of nonzero homogeneous elements of R. Then the following are equivalent:*

(1) *R is Schreier.*
(2) *R is pre-Schreier and R_0 is integrally closed in $(R_S)_0$.*
(3) *The homogeneous elements of R are primal and R_0 is integrally closed in $(R_S)_0$.*
(4) *R is gr-pre-Schreier and integrally closed.*

3. Pre-Schreier Monoid Domains

In this section we specialize to a monoid domain $R = A[M]$ where A is an integral domain and M is a nontrivial cancellative torsion-free commutative monoid. As we have already noted, for monoid domains the implications in (AA) are reversible. Further it is easy to show the following:

Lemma 3.1. *Let A be an integral domain and M a nontrivial cancellative torsion-free commutative monoid. Then $A[M]$ is gr-pre-Schreier if and only if A and M are pre-Schreier.*

Theorem 3.2. *Let A be an integral domain and M a nontrivial cancellative torsion-free commutative monoid. Then the following are equivalent:*

(1) *$A[M]$ is Schreier.*
(2) *$A[M]$ is pre-Schreier.*
(3) *A and M are Schreier.*

Proof. The implication (1) $\Rightarrow$ (2) is clear, and the converse is true since, by Theorem 2.1, $A[M]$ has property $(**)$, which, for a monoid ring, implies that $A[M]$ is integrally closed.

The implications (1) $\Leftrightarrow$ (3) follow easily from (1) $\Leftrightarrow$ (4) of Theorem 2.2, Lemma 3.1, and the fact that, by [11, 12.11], $A[M]$ is integrally closed if and only if A and M are integrally closed. □

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