Radio Number for Square Paths *

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Abstract

Let G be a connected graph. For any two vertices u and v, let $\mathrm{d}(u,v)$ denote the distance between u and v in G. The maximum distance between any pair of vertices is called the diameter of G and denoted by $\mathrm{diam}(G)$. A *radio-labeling* (or multi-level distance labeling) with span k for G is a function f that assigns to each vertex with a label from the set $\{0, 1, 2, \cdots, k\}$ such that the following holds for any vertices u and v: $|f(u) - f(v)| \geq \mathrm{diam}(G) - \mathrm{d}(u,v) + 1$. The *radio number* of G is the minimum span over all radio-labelings of G. The *square* of G is a graph constructed from G by adding edges between vertices of distance two apart in G. In this article, we completely determine the radio number for the square of any path.

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1 Introduction

Radio-labeling (cf. [3, 4]) is motivated by the channel assignment problem introduced by Hale [9]. Suppose we are given a set of stations or transmitters, the task is to assign to each station (or transmitter) with a channel (non-negative integer) such that the interference is avoided. The interference is closely related to the geographical locations of the stations – the closer are the stations the stronger the interference that might occur. To avoid interference, the separation of the channels assigned to near-by stations must be large enough. To model this problem, we construct a graph so that each station is represented by a vertex, and two vertices are adjacent when their corresponding stations are close. The ultimate goal is to find a *valid* labeling such that the span (range) of the channels used is minimized.

Let G be a connected graph. For any two vertices u and v, the *distance* between u and v, denoted by $\mathrm{d}_G(u,v)$ (or $\mathrm{d}(u,v)$ when G is understood in the context), is the length of a shortest $(u,v)-$path in G. A *distance-two labeling* with span k is a function, $f : V(G) \to \{0,1,2,\cdots,k\}$, such that the following are satisfied:

$$|f(u) - f(v)| \geq \begin{cases} 2, & \text{if } \mathrm{d}(u,v) = 1; \\ 1, & \text{if } \mathrm{d}(u,v) = 2. \end{cases}$$

The *λ-number* of G is the smallest k such that G admits a distance-two labeling with span k. Since introduced by Griggs and Yeh [8] in 1992, distance-two labeling has been studied extensively (cf. [1, 2, 5 - 8, 10, 11, 14, 15, 17, 18]).

Radio-labeling extends the number of interference level considered in distance-two labeling from two to the largest possible – the diameter of G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance among all pairs of vertices in G. A *radio-labeling* (or *multi-level distance labeling* [16, 12]) with span k for a graph G is a function, $f : V(G) \to \{0,1,2,\cdots,k\}$, such that the following holds for any u and v:

$$|f(u) - f(v)| \geq \mathrm{diam}(G) - \mathrm{d}(u,v) + 1.$$

The *radio number* of G, denoted by $rn(G)$, is the minimum span of a radio-labeling for G. Note that if $\mathrm{diam}(G) = 2$, then radio-labeling and distance-two labeling become identical.

The radio number for different families of graphs has been studied by several authors. The radio numbers for paths and cycles were investigated by Chartrand et al. [3], Chartrand, Erwin and Zhang [4], and Zhang [19], and were completely solved by Liu and Zhu [16]. The radio number for trees was investigated in [12].

The *square* of a graph G, denoted by G^2, is the graph constructed by adding to G with edges connecting pairs of vertices that are distance two apart in G. We call the square of a path (or cycle, respectively) a *square path* (or *square cycle*, respectively). The radio number for square cycles has been studied in [13], in which the exact values were determined for most of the square cycles, while bounds were given for others.

In this article, we completely determine the radio number for square paths.

Theorem 1 *Let P_n^2 be a square path on n vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) = \begin{cases} k^2 + 2, & \text{if } n \equiv 1 \pmod 4 \text{ and } n \geq 9; \\ k^2 + 1, & \text{otherwise.} \end{cases}$$

2 Lower Bound

In this section, we establish the lower bound for Theorem 1. Throughout the article, we denote a path with n vertices by P_n, where $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i = 1, 2, \cdots, n-1\}$. Hence, $V(P_n^2) = V(P_n)$ and $E(P_n^2) = E(P_n) \cup \{v_i v_{i+2} : i = 1, 2, 3, \cdots, n-2\}$. The diameter of P_n^2 is $\lfloor \frac{n}{2} \rfloor$.

We denote the distance between two vertices u and v in P_n^2 by $\mathrm{d}(u, v)$. Observe,

Proposition 2 *For any $u, v \in V(P_n^2)$, we have*

$$\mathrm{d}(u,v) = \lceil \frac{\mathrm{d}_{P_n}(u,v)}{2} \rceil.$$

A *center* of P_n is defined as a "middle" vertex of P_n. An odd path P_{2k+1} has only one center v_{k+1}, while an even path P_{2k} has two centers v_k and v_{k+1}. For each vertex $u \in V(P_n)$, the *level* of u, denoted by $L(u)$, is the smallest distance in P_n from u to a center of P_n. For instance, if $n = 2k+1$, then $L(v_1) = k$ and $L(v_{k+1}) = 0$. Denote the levels of a sequence of vertices A by $L(A)$. If $n = 2k+1$, then

$$L(v_1, v_2, \cdots, v_{2k+1}) = (k, k-1, \cdots, 3, 2, 1, 0, 1, 2, 3, \cdots, k-1, k)$$

If $n = 2k$, then

$$L(v_1, v_2, \cdots, v_{2k}) = (k-1, k-2, \cdots, 2, 1, 0, 0, 1, 2, \cdots, k-2, k-1).$$

Define the *left- and right-vertices* by: If $n = 2k+1$, then the left- and right-vertices, respectively, are

$$\{v_1, v_2, \cdots, v_k, v_{k+1}\} \text{ and } \{v_{k+1}, v_{k+2}, v_{k+3}, \cdots, v_{2k+1}\}.$$

(Note here the center v_{k+1} is both a left- and right-vertex). If $n = 2k$, then the left- and right-vertices, respectively, are

$$\{v_1, v_2, \cdots, v_k\} \text{ and } \{v_{k+1}, v_{k+2}, v_{k+3}, \cdots, v_{2k}\}.$$

If two vertices are both right (or left)-vertices, then we say they are on the same side; otherwise, they are on the opposite sides. Observe

Lemma 3 *If $n = 2k+1$, then for any $u, v \in V(P_n^2)$, we have:*

$$\mathrm{d}(u,v) = \begin{cases} \lceil \frac{L(u)+L(v)}{2} \rceil, & \text{if } u \text{ and } v \text{ are on the opposite sides;} \\ \lceil \frac{|L(u)-L(v)|}{2} \rceil, & \text{otherwise.} \end{cases}$$

If $n = 2k$, then for any $u, v \in V(P_n^2)$, we have:

$$\mathrm{d}(u,v) = \begin{cases} \lceil \frac{L(u)+L(v)+1}{2} \rceil, & \text{if } u \text{ and } v \text{ are on the opposite sides;} \\ \lceil \frac{|L(u)-L(v)|}{2} \rceil, & \text{otherwise.} \end{cases}$$

Lemma 4 *Let P_n^2 be a square path on n vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) \geq \begin{cases} k^2+2, & \text{if } n \equiv 1 \pmod 4 \text{ and } n \geq 9; \\ k^2+1, & \text{otherwise.} \end{cases}$$

Proof. Let f be a radio-labeling for P_n^2. Re-arrange $V(P_n^2) = \{x_1, x_2, \cdots, x_n\}$ with $0 = f(x_1) < f(x_2) < f(x_3) < \ldots < f(x_n)$. We first claim that $rn(P_n^2) \geq k^2+1$, by showing $f(x_n) \geq k^2+1$.

By definition, $f(x_{i+1}) - f(x_i) \geq k+1-\mathrm{d}(x_{i+1}, x_i)$ for $1 \leq i \leq n-1$. Summing up these $n-1$ in-equalities, we have

$$f(x_n) \geq (n-1)(k+1) - \sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1}).$$

Thus, to minimize $f(x_n)$ it bounds to maximize the sum $\sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1})$.

Assume $n = 2k$. By Lemma 3, we have

$$\sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{L(x_{i+1}) + L(x_i) + 1}{2} \rceil.$$

Observe from the above we have: 1) The equality holds only when x_i and x_{i+1}, for any i, are on the opposite sides unless one of them is a center, and 2) in the last summation term, each vertex of P_n^2 occurs exactly twice, except x_1 and x_n, for which each occurs only once. Note, $\lceil (L(x_{i+1}) + L(x_i) + 1)/2 \rceil \leq (L(x_{i+1}) + L(x_i) + 2)/2$, and the equality holds only if $L(x_{i+1})$ and $L(x_i)$ have the same parity. Combining this with 1), there exist at most $n-2$ of the i's such that $\mathrm{d}(x_i, x_{i+1}) = (L(x_{i+1}) + L(x_i) + 2)/2$. Moreover, among all the vertices only the two centers are of level 0, we conclude that

$$\sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1}) \leq 4 \sum_{i=1}^{k-1} \frac{i}{2} + \frac{n-1}{2} - \frac{0}{2} + \frac{n-2}{2} = k^2 + k - \frac{3}{2}.$$

Hence,

$$rn(P_n^2) \geq (2k-1)(k+1) - k^2 - k + 2 = k^2 + 1.$$

Assume $n = 2k + 1$. By Lemma 3, we have

$$\sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{L(x_{i+1}) + L(x_i)}{2} \rceil.$$

Similar to the above discussion, and since there is only one vertex with level 0, we conclude that

$$\sum_{i=1}^{n-1} \mathrm{d}(x_i, x_{i+1}) \leq 4 \sum_{i=1}^{k} \frac{i}{2} - \frac{1+0}{2} + \frac{n-2}{2} = k^2 + 2k - 1.$$

Hence, $rn(P_n^2) \geq 2k(k+1) - (k^2 + 2k - 1) = k^2 + 1$.

Assume $n \equiv 1 \pmod 4$ and $n \geq 9$, that is, n is odd, k is even, and $k \geq 4$. Assume to the contrary that $f(x_n) = k^2 + 1$. Then by the above discussion, all the following must hold:

1) $\{x_1, x_n\} = \{v_{k+1}, v_k\}$ or $\{x_1, x_n\} = \{v_{k+1}, v_{k+2}\}$.

2) $f(x_{i+1}) = f(x_i) + k + 1 - \mathrm{d}(x_{i+1}, x_i)$ for all i.

3) For any $i \geq 1$, the two vertices x_i and x_{i+1} belong to opposite sides unless one of them is the center.

4) There exists some $1 \leq t \leq n-1$ such that $L(x_t) \equiv L(x_{t+1}) \pmod 2$, while $L(x_i) \not\equiv L(x_{i+1}) \pmod 2$ for all other $i \neq t$.

By 1) and by symmetry, we may assume $x_1 = v_{k+1}$. Since k is even, $k \geq 4$, and v_{k+1} is the only center of P_{2k+1}^2, by 1) - 4), we obtain:

5) $L(x_1)$, $L(x_3)$, $L(x_5)$, $\cdots$, $L(x_{k+1})$, $L(x_{k+2})$, $L(x_{k+4})$, $\cdots$, $L(x_{2k})$ are all even, while the levels of other vertices are all odd.

Claim. $\{v_1, v_{2k+1}\} = \{x_{k+1}, x_{k+2}\}$.

Proof) Suppose $v_1 \notin \{x_{k+1}, x_{k+2}\}$. As $L(v_1) = k$, by 5), we have $v_1 = x_a$ for some a where x_{a-1} and x_{a+1} are vertices on the right side and both $L(x_{a-1})$ and $L(x_{a+1})$ are odd. Let $L(x_{a-1}) = y$ and $L(x_{a+1}) = z$. By 2),

$f(x_a) - f(x_{a-1}) = k+1-(k+y+1)/2$ and $f(x_{a+1}) - f(x_a) = k+1-(k+z+1)/2$. This implies $f(x_{a+1}) - f(x_{a-1}) = k+1-(y+z)/2$, contradicting that $f(x_{a+1}) - f(x_{a-1}) \geq k+1-|y-z|/2$ (as z, y are odd so $z, y \neq 0$). Therefore, $v_1 \in \{x_{k+1}, x_{k+2}\}$. Similarly, one can show that $v_{2k+1} \in \{x_{k+1}, x_{k+2}\}$. ∎

By the Claim, we may assume that $v_1 = x_{k+1}$ and $v_{2k+1} = x_{k+2}$ (the proof for the other case is symmetric). By 5), $L(x_{k+3}) = b$ for some odd b. By 3) and 2), we have $f(v_{2k+1}) - f(v_1) = 1$ and $f(x_{k+3}) - f(v_{2k+1}) = k+1-(b+k+1)/2$. So, $f(x_{k+3}) - f(v_1) = (k-b+3)/2$. By definition and Lemma 3, $f(x_{k+3}) - f(v_1) \geq k+1-(k-b+1)/2$. Hence, we have $b = 1 = L(x_{k+3})$. This implies that $x_{k+3} = v_k$. Similarly, we can get $x_k = v_{k+2}$. By 1), it must be $k = 2$, contradicting the assumption $k \geq 4$. Therefore, $rn(P^2_{2k+1}) \geq k^2 + 2$. ∎

3 Upper Bound and Optimal Radio-Labelings

By Lemma 4, to establish Theorem 1, it suffices to give radio-labelings achieving the desired spans. To this end, we will use the following lemma.

Lemma 5 *Let P^2_n be a square path on n vertices with $k = \lfloor n/2 \rfloor$. Let $\{x_1, x_2, \cdots, x_n\}$ be a permutation of $V(P^2_n)$ such that for any $1 \leq i \leq n-2$,*

$$\min\{\mathrm{d}_{P_n}(x_i, x_{i+1}), \mathrm{d}_{P_n}(x_{i+1}, x_{i+2})\} \leq k+1,$$

and if k is even and the equality in the above holds, then $\mathrm{d}_{P_n}(x_i, x_{i+1})$ and $\mathrm{d}_{P_n}(x_{i+1}, x_{i+2})$ have different parities. Let f be a function, $f : V(P^2_n) \to \{0, 1, 2, \cdots\}$ with $f(x_1) = 0$, and $f(x_{i+1}) - f(x_i) = k+1-\mathrm{d}(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio-labeling for P^2_n.

Proof. Recall, $\mathrm{diam}(P^2_n) = k$. Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k+1-\mathrm{d}(x_i, x_j)$ for any $j \geq i+2$. For $i = 1, 2, \cdots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

For any $j \geq i+2$, it follows that

$$f(x_j) - f(x_i) = f_i + f_{i+1} + \cdots + f_{j-1}.$$

Suppose $j = i+2$. Assume $\mathrm{d}(x_i, x_{i+1}) \geq \mathrm{d}(x_{i+1}, x_{i+2})$. (The proof for $\mathrm{d}(x_{i+1}, x_{i+2}) \geq \mathrm{d}(x_i, x_{i+1})$ is similar.) Then, $\mathrm{d}(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. Let $x_i = v_a$, $x_{i+1} = v_b$, and $x_{i+2} = v_c$. It suffices to consider the following cases:

- $\underline{b < a < c \ \text{ or } \ c < a < b}$. By the assumption $\mathrm{d}(x_i, x_{i+1}) \geq \mathrm{d}(x_{i+1}, x_{i+2})$, we obtain $\mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$ and $\mathrm{d}(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k+1-\mathrm{d}(x_i, x_{i+1}) + k + 1 - \mathrm{d}(x_{i+1}, x_{i+2}) \\ &\geq 2k+2-2\left(\frac{k+2}{2}\right) \\ &= k+1-\mathrm{d}(x_i, x_{i+2}). \end{aligned}$$

- $\underline{a < b < c \ \text{ or } \ c < b < a}$. Then $\mathrm{d}(x_i, x_{i+2}) \geq \mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, x_{i+2}) - 1$. Similar to the above, easy calculation shows that $f(x_{i+2}) - f(x_i) \geq k+1-\mathrm{d}(x_i, x_{i+2})$.

- $\underline{a < c < b \ \text{ or } \ b < c < a}$. Assume k is odd or $\min\{\mathrm{d}_{P_n}(x_i, x_{i+1}), \mathrm{d}_{P_n}(x_{i+1}, x_{i+2})\} \leq k$, then we have $\mathrm{d}(x_{i+1}, x_{i+2}) \leq (k+1)/2$ and $\mathrm{d}(x_i, x_{i+2}) \geq \mathrm{d}(x_i, x_{i+1}) - \mathrm{d}(x_{i+1}, x_{i+2})$. Hence, $f(x_{i+2}) - f(x_i) \geq k+1-\mathrm{d}(x_i, x_{i+2})$.

  If k is even and $\min\{\mathrm{d}_{P_n}(x_i, x_{i+1}), \mathrm{d}_{P_n}(x_{i+1}, x_{i+2})\} = k+1$, then by our assumption, it must be that $\mathrm{d}_{P_n}(x_{i+1}, x_{i+2}) = k+1$ and $\mathrm{d}_{P_n}(x_i, x_{i+1})$ is even. Hence, we have $\mathrm{d}(x_i, x_{i+2}) = \mathrm{d}(x_i, x_{i+1}) - \mathrm{d}(x_{i+1}, x_{i+2}) + 1$. This implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &\geq 2k+2-2\mathrm{d}(x_{i+1}, x_{i+2}) - \mathrm{d}(x_i, x_{i+2}) + 1 \\ &= 2k+2-2\left(\tfrac{k+2}{2}\right) - \mathrm{d}(x_i, x_{i+2}) + 1 \\ &= k+1-\mathrm{d}(x_i, x_{i+2}). \end{aligned}$$

Let $j = i+3$. First, we assume that the sum of some pair of the distances $\mathrm{d}(x_i, x_{i+1})$, $\mathrm{d}(x_{i+1}, x_{i+2})$, $\mathrm{d}(x_{i+2}, x_{i+3})$ is at most $k+2$. Then

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k+3 - \mathrm{d}(x_i, x_{i+1}) - \mathrm{d}(x_{i+1}, x_{i+2}) - \mathrm{d}(x_{i+2}, x_{i+3}) \\ &\geq k+1 \ > \ k+1 - \mathrm{d}(x_i, x_{i+3}). \end{aligned}$$

Next, we assume that the sum of every pair of the distances $\mathrm{d}(x_i, x_{i+1})$, $\mathrm{d}(x_{i+1}, x_{i+2})$ and $\mathrm{d}(x_{i+2}, x_{i+3})$ is greater than $k+2$. Then, by our hypotheses it follows that

$$\mathrm{d}(x_i, x_{i+1}), \mathrm{d}(x_{i+2}, x_{i+3}) > \frac{k+2}{2} \ \text{ and } \ \mathrm{d}(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}. \qquad (*)$$

Let $x_i = v_a$, $x_{i+1} = v_b$, $x_{i+2} = v_c$, $x_{i+3} = v_d$. Because $\mathrm{diam}(P_n^2) = k$, by (*) and our assumption that the sum of any pair of the distances, $\mathrm{d}(x_i, x_{i+1})$, $\mathrm{d}(x_{i+1}, x_{i+2})$, $\mathrm{d}(x_{i+2}, x_{i+3})$, is great than $k+2$, it must be that

- $\underline{a < c < b < d \ \ (\text{or} \ \ d < b < c < a)}$. Then $d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1$. By (*), we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k+3 - \mathrm{d}(x_i, x_{i+1}) - \mathrm{d}(x_{i+1}, x_{i+2}) - \mathrm{d}(x_{i+2}, x_{i+3}) \\ &\geq 3k+2 - 2\mathrm{d}(x_{i+1}, x_{i+2}) - \mathrm{d}(x_i, x_{i+3}) \\ &\geq k+1 - \mathrm{d}(x_i, x_{i+3}). \end{aligned}$$

Let $j \geq i+4$. Since $\min\{\mathrm{d}(x_i, x_{i+1}), \mathrm{d}(x_{i+1}, x_{i+2})\} \leq \frac{k+2}{2}$, and $f_i \geq k+1-\mathrm{d}(x_i, x_{i+1})$ for any i, we have $\max\{f_i, \ f_{i+1}\} \geq \frac{k}{2}$ for any $1 \leq i \leq n-2$. Hence, $f(x_j) - f(x_i) \geq f_i + f_{i+1} + f_{i+2} + f_{i+3} > k+1 > k+1-d(x_i, x_j)$. ∎

To show the existence of a radio-labeling achieving the desired bound, we consider cases separately. For each radio-labeling f given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^2) = \{x_1, x_2, \cdots, x_n\}$, then define f by $f(x_1) = 0$ and for $i = 1, 2 \cdots, n-1$:

$$f(x_{i+1}) = f(x_i) + k + 1 - \mathrm{d}(x_i, x_{i+1}).$$

Case 1: $rn(P^2_{2k+1}) \leq k^2+1$, if k is odd. We give a radio-labeling with span k^2+1. The line-up (permutation) of $V(P^2_{2k+1}) = \{x_1, x_2, x_3, \cdots, x_{2k+1}\}$ is given by the arrows in Table 1. That is, $x_1 = v_k$, $x_2 = v_{2k}$, $\cdots$, $x_{2k+1} = v_{k+1}$. The value above each arrow shows the distance between the two consecutive vertices in P_n.

$$v_k \xrightarrow{k} v_{2k} \xrightarrow{k+2} v_{k-2} \xrightarrow{k} v_{2k-2} \xrightarrow{k+2} \cdots \xrightarrow{k} v_{k+3} \xrightarrow{k+2} v_1$$
$$\xrightarrow{k+1} v_{k+2} \xrightarrow{k} v_2 \xrightarrow{k+2} v_{k+4} \xrightarrow{k} v_4 \xrightarrow{k+2} v_{k+6} \xrightarrow{k} \cdots \xrightarrow{k+2} v_{2k+1} \xrightarrow{k} v_{k+1}.$$

Table 1: Vertex ordering of a labeling for P^2_{2k+1} for odd k.

By Lemma 5, f is a radio-labeling for P^2_n. As k is odd, observe from Table 1, there are two possible distances in P^2_{2k+1} between consecutive vertices, namely, $\frac{k+1}{2}$ and $\frac{k+3}{2}$, with the number of occurences $k+1$ and $k-1$, respectively. It follows by direct calculation that $f(x_{2k+1}) = 2k(k+1) - \sum_{i=1}^{2k} \mathrm{d}(x_i, x_{i+1}) = k^2+1$. As an example, the following is an optimal radio-labeling (with minimum span) for P^2_{15}:

21 29 14 36 7 43 0 50 25 18 32 11 39 4 46.

Case 2: $rn(P^2_{2k}) \leq k^2+1$, if k is odd. Let $G = P^2_{2k+1}$ for some odd k. Let H be the subgraph of G induced by the vertices $\{v_1, v_2, \cdots, v_{2k}\}$. Then $H \cong P^2_{2k}$, $\mathrm{diam}(H) = \mathrm{diam}(G) = k$, and $\mathrm{d}_G(u,v) = \mathrm{d}_H(u,v)$ for every $u, v \in V(H)$. Let f be a radio-labeling for G. Then f restricted to H is a radio-labeling for H. By Case 1, $rn(P^2_{2k}) \leq rn(P^2_{2k+1}) \leq k^2+1$.

Case 3: $rn(P^2_{2k}) \leq k^2+1$, if k is even. Similar to Case 1, we line-up the vertices according to Table 2.

By Lemma 5, f is a radio-labeling for P^2_n. Indeed, observe from Table 2, as k is even, there are three possible distances between any consecutive

$$v_k \xrightarrow{k-1} v_{2k-1} \xrightarrow{k+1} v_{k-2} \xrightarrow{k+2} v_{2k} \xrightarrow{k+1} v_{k-1} \xrightarrow{k-1} v_{2k-2} \xrightarrow{k+1} v_{k-3} \cdots \xrightarrow{k+1} v_1$$

$$\xrightarrow{2k-4} v_{2k-3} \xrightarrow{k+1} v_{k-4} \xrightarrow{k-1} v_{2k-5} \xrightarrow{k+1} v_{k-6} \xrightarrow{k-1} \cdots \xrightarrow{k+1} v_2 \xrightarrow{k-1} v_{k+1}.$$

Table 2: Vertex ordering of a labeling for P_{2k}^2, for some even k.

vertices in P_{2k}^2, namely, $\frac{k}{2}$, $\frac{k+2}{2}$ and $k-2$, with the number of occurences $k-2$, k and 1, respectively. By some calculation, the span of f is k^2+1. As an example, the following is an optimal radio-labeling for P_{16}^2:

44 60 35 51 26 9 17 0 65 40 56 31 47 22 5 13.

We now consider the case $n = 2k+1$ for even k. If $k = 2$, the labeling $(3, 5, 0, 2, 4)$ is a radio-labeling for P_5^2. Hence, we assume $k \geq 4$.

Case 4: If k is even and $k \geq 4$, then $rn(P_{2k+1}^2) \leq k^2+2$. Similarly, we arrange the vertices according to Table 3.

$$v_k \xrightarrow{k+1} v_{2k+1} \xrightarrow{2k} v_1 \xrightarrow{k+1} v_{k+2} \xrightarrow{k-1} v_3 \xrightarrow{k+1} v_{k+4} \cdots \xrightarrow{k-1} v_{k-1} \xrightarrow{k+1} v_{2k}$$

$$\xrightarrow{2k-2} v_2 \xrightarrow{k+1} v_{k+3} \xrightarrow{k-1} v_4 \xrightarrow{k+1} v_{k+5} \xrightarrow{k-1} v_6 \xrightarrow{k+1} \cdots \xrightarrow{k-1} v_{k-2} \xrightarrow{k+1} v_{2k-1} \xrightarrow{k-2} v_{k+1}$$

Table 3: Vertex ordering of a labeling for P_{2k+1}^2, for even k.

By Lemma 5, f is a radio-labeling for P_{2k+1}^2. Observe from Table 3, there are five possible distances between any consecutive vertices x_i and x_{i+1} in P_n^2, namely, k, $k-1$, $\frac{k-2}{2}$, $\frac{k+2}{2}$ and $\frac{k}{2}$, with the number of occurences 1, 1, 1,

k and $k-3$, respectively. By some calculation, one can show that the span of f is k^2+2. As an example, the following is an optimal radio-labeling for P_{17}^2:

5 38 14 47 23 56 32 0 66 9 42 18 51 27 60 36 4.

This completes the proof of Theorem 1.

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