Radio Number for Trees

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Abstract

Let G be a connected graph with diameter $\mathrm{diam}(G)$. The *radio number* for G, denoted by $\mathrm{rn}(G)$, is the smallest integer k such that there exists a function $f : V(G) \to \{0, 1, 2, \cdots, k\}$ with the following satisfied for all vertices u and v: $|f(u)-f(v)| \geq \mathrm{diam}(G)-d_G(u, v)+1$, where $d_G(u, v)$ is the distancee between u and v. We prove a lower bound for the radio number of trees, and characterize the trees achieving this bound. Moreover, we prove another lower bound for the radio number of spiders (trees with at most one vertex of degree more than two) and characterize the spiders achieving this bound. Our results generalize the radio number for paths obtained by Liu and Zhu.

1 Introduction

Multi-level distance labeling (or radio labeling) can be regarded as an extension of distance-two labeling, and both of them are motivated by the channel assignment problem introduced by Hale [8]. Given a set of stations (or transmitters), a *valid channel assignment* is a function that assigns to each station

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with a channel (nonnegative integer) such that interference is avoided. The task is to find a valid channel assignment with the minimum span of the channels used. The degree (or level) of interference is related to the locations of the stations – the closer of two stations, the stronger interference that might occur. In order to avoid interference, the separation between the channels assigned to a pair of near-by stations must be large enough; the amount of the required separation depends on the distance between the two stations.

A graph model for this problem is to represent each station by a vertex, and connect every pair of close stations by an edge. Let G be a connected graph. We denote the distance between two vertices u and v by $d_G(u, v)$, or $d(u, v)$ if G is clear in the context. Motivated by the channel assignment problem with two levels of interference, a *distance-two labeling* for G is a function $f : V \to \{0, 1, 2, 3, \cdots\}$ such that $|f(u)-f(v)| \geq 2$ if $d(u, v) = 1$; and $f(u) \neq f(v)$ if $d(u, v) = 2$. The *span* of f is defined as $\max_{u,v \in V}\{f(u)-f(v)\}$. The λ-number for a graph G, denoted by $\lambda(G)$, is the minimum span of a distance-two labeling for G. Distance-two labeling has been studied intensively in the past decade (cf. [1, 2, 5 - 7, 9 - 11, 16]).

Motivated by the channel assignment problem with diam(G) levels of interference, a *multi-level distance labeling* (or *radio labeling*) is a function $f : V(G) \to \{0, 1, 2, 3, \cdots\}$ so that the following is satisfied for $u, v \in V(G)$:

$$|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1,$$

where diam(G) is the *diameter* of G (the maximum distance over all pairs of vertices). The *radio number* (as suggested by the FM radio channel assignment [4]) for a graph G, denoted by rn(G), is the minimum span of a radio labeling for G. Note that when diam$(G) = 2$, distance-two labeling coincides with radio labeling, and in this case, $\lambda(G) = \text{rn}(G)$.

Finding the radio number for a graph is an interesting yet challenging task. So far, the value is known only to very limited families of graphs. For

paths and cycles, it was studied by Charchand et al. [4, 3] and Zhang [17], while the exact value remained open until lately solved by Liu and Zhu [13]. The radio number for square paths (adding edges between vertices of distance two apart) was determined by Liu and Xie [14] who also studied the problem for square cycles [15].

The aim of this article is to extend the study to trees. In Section 2, we prove a general lower bound for the radio number of trees and characterize the trees achieving this bound. Then we focus on the study of a special family of trees called *spiders* which are trees with at most one vertex of degree more than two. Besides the lower bound obtained by applying the result of trees to spiders, in Section 3, we present another lower bound for spiders and characterize the spiders achieving the bounds.

2 A Lower Bound for Trees

As we are seeking for the minimum span of a radio labeling for a graph G, without loss of generality, we always assume that the label 0 is used by any radio labeling f. So the span of f is the maximum label used. A radio labeling for G with span equal to $\mathrm{rn}(G)$ is called an *optimal radio labeling*.

Let T be a tree rooted at a vertex w. For any two vertices u and v, if u is on the (w, v)-path, then u is an *ancestor* of v, and v is a *descendent* of u. The root w is an ancestor of every vertex, and every vertex is its own ancestor and descendent. Fix any w as the root, define the *level function* on $V(T)$ by

$$L_w(u) = d(w, u), \text{ for any } u \in V(T).$$

For any $u, v \in V(T)$, define

$$\phi_w(u, v) = \max \{L_w(t) : t \text{ is a common ancestor of } u \text{ and } v\}.$$

Let w' be a neighbor of w. We call the subtree induced by w' together with all the descendents of w' a *branch*.

Observation 1 *Let T be a tree rooted at w. For any vertices u and v,*

(1) $\phi_w(u,v) = 0$ if and only if u and v belong to different branches (unless one of them is w), and

(2) $d(u,v) = L_w(u) + L_w(v) - 2\phi_w(u,v)$.

For any vertex w in a tree T, the *weight of T (rooted) at w* is defined by:

$$w_T(w) = \sum_{u \in V(T)} L_w(u).$$

The *weight of T* is the smallest weight among all vertices of T:

$$w(T) = \min \{w_T(w) : w \in V(T)\}.$$

A vertex w^* of a tree T is called a *weight center* of T if $w_T(w^*) = w(T)$.

If ww' is an edge of T and $T_w, T_{w'}$ are two components of $T - ww'$, then it follows easily from the definition that $\omega_T(w) = \omega_T(w') + |V(T_{w'})| - |V(T_w)|$. Therefore, the next two lemmas emerge.

Lemma 1 *Suppose w^* is a weight center of a tree T. Then each component of $T - w^*$ contains at most $|V(T)|/2$ vertices.*

Lemma 2 *Every tree T has either one or two weight centers, and T has two weight centers, say w and w', if and only if ww' is an edge of T and $T - ww'$ consists of two equal-sized components.*

A radio labeling is a one-to-one function. On the other hand, any one-to-one integral function f on $V(G)$, with $0 \in f(V)$, induces an *ordering* of $V(G)$, which is a line-up of the vertices with increasing images. We denote this ordering by $U(f)$, where $V(G) = U(f) = \{u_0, u_1, u_2, \cdots, u_{|V|-1}\}$ with

$$0 = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{|V|-1}).$$

Notice, if f is a radio labeling, then the span of f is $f(u_{|V|-1})$.

Theorem 3 *Let T be an m-vertex tree with diameter d. Then*

$$\mathrm{rn}(T) \geq (m-1)(d+1) + 1 - 2w(T).$$

Moreover, the equality holds if and only if for every weight center w^*, there exists a radio labeling f with $f(u_0) = 0 < f(u_1) < \cdots < f(u_{m-1})$, where all the following hold (for all $0 \leq i \leq m-2$):

(1) u_i and u_{i+1} belong to different branches (unless one of them is w^*);

(2) $\{u_0, u_{m-1}\} = \{w^*, v\}$, where v is some vertex with $L_{w^*}(v) = 1$;

(3) $f(u_{i+1}) = f(u_i) + d + 1 - L_{w^*}(u_i) - L_{w^*}(u_{i+1})$.

Proof. Let f be an optimal radio labeling for T, where $f(u_0) = 0 < f(u_1) < f(u_2) < \cdots < f(u_{m-1})$. Then $f(u_{i+1}) - f(u_i) \geq (d+1) - d(u_{i+1}, u_i)$ for all $0 \leq i \leq m-2$. Summing up these $m-1$ inequalities, we get

$$\mathrm{rn}(T) = f(u_{m-1}) \geq (m-1)(d+1) - \sum_{i=0}^{m-2} d(u_{i+1}, u_i). \tag{2.1}$$

Let w^* be a weight center. Each vertex of T occurs exactly twice in the last summation in (2.1), except u_0 and u_{m-1}, for which each occurs exactly once. Hence, by Observation 1, we get

$$\begin{aligned} \sum_{i=0}^{m-2} d(u_{i+1}, u_i) &= 2\left(\sum_{u \in V(T)} L_{w^*}(u)\right) - L_{w^*}(u_0) - L_{w^*}(u_{m-1}) - 2\sum_{i=0}^{m-2} \phi_{w^*}(u_{i+1}, u_i) \\ &\leq 2\left(\sum_{u \in V(T)} L_{w^*}(u)\right) - 1 = 2w(T) - 1. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), the lower bound for $\mathrm{rn}(T)$ is obtained.

The equality in (2.2) holds if and only if $\phi_{w^*}(u_{i+1}, u_i) = 0$ for all i, and $\{u_0, u_{m-2}\} = \{w^*, v\}$ for some v with $L_{w^*}(v) = 1$. Combining this with (2.1), we derive one direction of the moreover part. To prove the converse, let w^* be a weight center. Suppose there exists a radio labeling f such that (1 - 3) hold. By (2) and (3), $f(u_{m-1}) = (m-1)(d+1) + 1 - 2\sum_{u \in V(T)} L_{w^*}(u) = (m-1)(d+1) + 1 - 2w(T)$. ∎

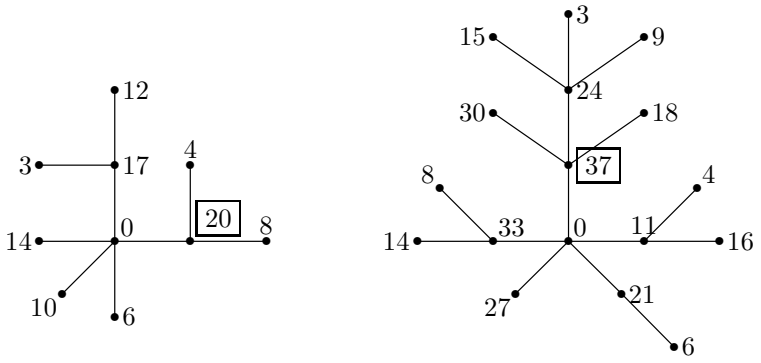


Figure 1: Optimal radio labelings for trees with radio numbers achieving the bound of Theorem 3.

Consequences of Theorem 3 include the radio number for paths (which was settled in [13] by a different approach). The radio number for P_{2k+1} is larger than the bound shown in Theorem 3, since there does not exist a radio labeling f that satisfies Theorem 3. It is not hard to find a radio labeling for P_{2k+1} with span one more than the bound of Theorem 3 (cf. [13]), hence $\mathrm{rn}(P_{2k+1})$ is obtained. Even paths P_{2k} have radio numbers equal to the bound in Theorem 3, as one can find a radio labeling satisfying Theorem 3 (cf. [13]).

Other than the even paths, there are many trees whose radio numbers achieve the bound in Theorem 3. See Figure 1 for examples.

3 Radio Number for Spiders

A spider with every vertex of degree at most two is indeed a path. As discussed in the previous section, the radio number for paths has been completely settled. Hence, we focus on the spiders with a vertex of degree more than two; we denote such a vertex by $v_{0,0}$. Notice that the methods used in this article can be extended to paths without difficulty.

We denote a spider by

$$S_{l_1,l_2,l_3,\cdots,l_n}, \text{ with } l_1 \geq l_2 \geq \cdots \geq l_n,\ n \geq 3,$$

where $l_i \in Z^+$ is the length of the i-th *leg* (a path with one end at $v_{0,0}$ and the other at an end-vertex). Hence, $|V(S_{l_1,l_2,\cdots,l_n})| = l_1 + l_2 + \cdots + l_n + 1$ and $\mathrm{diam}(S_{l_1,l_2,\cdots,l_n}) = l_1 + l_2$.

Let G be a spider, $G = S_{l_1,l_2,\cdots,l_n}$, $n \geq 3$. The vertex set of G is denoted by $V(G) = V_1 \cup V_2 \cdots \cup V_n$, where each V_i is the vertex set of the i-th leg, that is, assuming $v_{i,0} = v_{0,0}$,

$$V_i = \{v_{i,j} : 0 \leq j \leq l_i\}, \text{ where } \{v_{i,j}, v_{i,j+1}\} \in E(G),\ 0 \leq j \leq l_i - 1.$$

The level function for G (rooted) at $v_{0,0}$ is denoted by L. That is, $L(v_{i,j}) = j$. Notice that $v_{0,0}$ is not always a weight center. By Lemma 2, $v_{0,1}$ is a weight center if and only if $l_1 \leq |V(G)|/2$.

Throughout the section, we denote $\overline{l_1} = l_2 + l_3 + \cdots + l_n$.

Theorem 4 *Let $G = S_{l_1,l_2,\cdots,l_n}$ be a spider. Then*

$$w(G) = \begin{cases} \frac{1}{2}\sum l_i(l_i+1), & \text{if } l_1 \leq \overline{l_1} + 1; \\ \frac{1}{2}\sum_{i=1}^{n} l_i(l_i+1) - \lfloor \frac{l_1 - \overline{l_1}+1}{2} \rfloor \lceil \frac{l_1 - \overline{l_1}-1}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $m = |V(G)| = l_1 + l_2 + \cdots + l_n + 1$. By definition, $w_G(v_{0,0}) = \frac{1}{2}\sum_{i=1}^{n} l_i(l_i+1)$. By Lemma 1, $v_{0,0}$ is a weight center of G if and only if $l_1 \leq m/2$. This proves the case that $l_1 \leq \overline{l_1} + 1$.

Assume $l_1 > \overline{l_1} + 1$. By Lemmas 1 and 2, we may assume that a weight center of G is $v_{1,k}$, where $k = l_1 - \lfloor m/2 \rfloor$. Then

$$w(G) = w_G(v_{1,k}) = w_G(v_{0,0}) - k(l_1 - k) + k(m - l_1 - 1).$$

The result then follows by some calculation. ∎

By Theorems 3 and 4, we obtain

Corollary 5 *Let $G = S_{l_1,l_2,\cdots,l_n}$ be a spider. Then*

$$\mathrm{rn}(G) \geq \begin{cases} \sum_{i=1}^{n} l_i(l_1+l_2-l_i)+1, & \text{if } l_1 \leq \overline{l_1}+1; \\ \sum_{i=1}^{n} l_i(l_1+l_2-l_i)+1+2\lfloor \frac{l_1-\overline{l_1}+1}{2} \rfloor \lceil \frac{l_1-\overline{l_1}-1}{2} \rceil, & \text{otherwise.} \end{cases}$$

With the following few results, we establish anothor lower bound for spiders (Theorem 11), which in some cases, is better than the one in Corollary 5.

Observation 2 *The distance between any two vertices in $S_{l_1,l_2,\cdots,l_n}$ is*

$$d(v_{i,j}, v_{i',j'}) = \begin{cases} j+j' & \text{if } i \neq i'; \\ |j-j'| & \text{if } i = i'. \end{cases}$$

For a radio labeling f, we adopt the same notation from the previous section, $U(f) = \{u_0, u_1, \cdots, u_{|V|-1}\}$ with $0 = f(u_0) < f(u_1) < \cdots < f(u_{|V|-1})$. For any $0 \leq i \leq |V|-2$, set

$$x_i = f(u_{i+1}) - f(u_i) + L(u_{i+1}) + L(u_i) - \mathrm{diam}(G) - 1.$$

By Observation 2 and definition of radio labeling, $x_i \geq 0$ for any $0 \leq i \leq |V|-2$. Moreover, if $u_{i+1}, u_i \in V_k$ for some k, then $x_i \geq 2\min\{L(u_{i+1}), L(u_i)\}$.

For integers $0 \leq i < j \leq |V|-1$, the vertices $\{u_i,\ u_{i+1},\ u_{i+2},\ \cdots,\ u_j\}$ (respectively, the labels $\{f(u_i),\ f(u_{i+1}),\ \cdots,\ f(u_j)\}$) are called *consecutive vertices* (respectively, *consecutive labels*).

Lemma 6 *Let $G = S_{l_1,l_2,\cdots,l_n}$. Suppose f is a nonnegative integral one-to-one function on $V(G)$ with the ordering of $V(G) = U(f) = (u_0,\ u_1, u_2,\ \cdots,\ u_{|V|-1})$. Then f is a radio labeling for G if and only if the following hold for any set of consecutive vertices $\{u_i, u_{i+1}, u_{i+2}, \cdots, u_j\}$, $0 \leq i < j \leq |V|-1$:*

(1)
$$\sum_{t=i}^{j-1} x_t \geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1).$$

(2) If $u_i, u_j \in V_k$ for some k, then

$$\sum_{t=i}^{j-1} x_t \geq 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - (j-i-1)(l_1+l_2+1) + 2\min\{L(u_i), L(u_j)\}.$$

Proof. Suppose f is a radio labeling for G. Since $\mathrm{diam}(G) = l_1+l_2$, summing up x_t for $i \leq t \leq j-1$, we get

$$\sum_{t=i}^{j-1} x_t = f(u_j) - f(u_i) - (j-i)(l_1+l_2+1) + 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) + L(u_i) + L(u_j).$$

By Observation 2, $f(u_j) - f(u_i) \geq l_1 + l_2 + 1 - L(u_j) - L(u_i)$. So (1) holds.

To prove (2), again by definition and Observation 2, we have

$$\begin{aligned} f(u_j) - f(u_i) &= (j-i)(l_1+l_2+1) - 2\left(\sum_{t=i+1}^{j-1} L(u_t)\right) - L(u_i) - L(u_j) + \sum_{t=i}^{j-1} x_t \\ &\geq l_1 + l_2 + 1 - L(u_j) - L(u_i) + 2\min\{L(u_j), L(u_i)\}. \end{aligned}$$

Hence, (2) follows by easy calculation.

To prove the converse, assume f satisfies (1) and (2). To show that f is a radio labeling, it suffices to verify the following inequality for any $0 \leq i < j \leq |V| - 1$,

$$f(u_j) - f(u_i) \geq l_1 + l_2 + 1 - d(u_i, u_j). \tag{3.1}$$

If u_i and u_j belong to different legs, then $d(u_i, u_j) = L(u_i) + L(u_j)$. By (1) and Observation 2, (3.1) holds. If $u_i, u_j \in V_k$ for some k, then (3.1) follows by (2) and Observation 2. ∎

We introduce a few more notations. For a spider $G = S_{l_1,l_2,\cdots,l_n}$ with $l_1 - l_2 \geq 2$, let

$$z = \lfloor \frac{l_1 - l_2 - 2}{2} \rfloor.$$

Suppose f is a radio labeling for a spider with $l_1 - l_2 \geq 2$. For $j = 0, 1, 2, \cdots, z$, let t_j be the integers, $0 \leq t_j \leq |V| - 1$, with

$$u_{t_j} = v_{1,l_1-j}.$$

Lemma 7 *Let f be a radio labeling for $G = S_{l_1,l_2,\cdots,l_n}$, where $l_1 - l_2 \geq 2$. Let t_j, $j = 0, 1, 2, \cdots, z$, be defined as in the above. If $1 \leq t_j \leq |V| - 2$ for some $j = 0, 1, \cdots, z$, then $x_{t_j-1} + x_{t_j} \geq l_1 - l_2 - (2j + 1) \geq 1$. Moreover, the first inequality is strict when $u_{t_j-1}, u_{t_j+1} \in V_k - \{v_{0,0}\}$ for some k.*

Proof. Let $j = 0, 1, 2, \cdots, z$. Assume $v_{1,l_1-j} = u_{t_j}$ for some $1 \leq t_j \leq |V| - 2$. Consider the three consecutive vertices $\{u_{t_j-1}, u_{t_j}, u_{t_j+1}\}$. Since $L(u_{t_j}) = l_1 - j$, by Lemma 6 (1), we have

$$\begin{aligned} x_{t_j} + x_{t_j-1} &\geq 2L(u_{t_j}) - (l_1 + l_2 + 1) \\ &= l_1 - l_2 - 2j - 1 \geq 1. \end{aligned}$$

The last inequality is derived from $0 \leq j \leq \lfloor \frac{l_1-l_2-2}{2} \rfloor$ and $l_1 - l_2 \geq 2$.

To prove the moreover part, assume $u_{t_j-1}, u_{t_j+1} \in V_k - \{v_{0,0}\}$ for some k. By Lemma 6 (2), $x_{t_j} + x_{t_j-1} \geq 2L(u_{t_j}) - (l_1 + l_2 + 1) + 2 > l_1 - l_2 - 2j - 1$. ∎

Lemma 8 *Let f be a radio labeling for $G = S_{l_1,l_2,\cdots,l_n}$. If there exist some $0 \leq j, j' \leq z$, $j \neq j'$, such that $t_{j'} = t_j + 1$ (that is, v_{1,l_1-j} and $v_{1,l_1-j'}$ are consecutive), then $x_{t_j} > 2(l_1 - l_2 - j' - j - 1) = 2(l_1 - l_2) - (2j' + 1) - (2j + 1)$.*

Proof. By Lemma 6 (2), $x_{t_j} \geq 2 \min\{l_1 - j, l_1 - j'\} > 2(l_1 - l_2 - j' - j - 1)$. ∎

Lemma 9 *Let f be a radio labeling for $G = S_{l_1,l_2,\cdots,l_n}$. If $l_1 - l_2 \leq 1$, or $l_1 - l_2 \geq 2$ and $1 \leq t_j \leq |V| - 2$ for all $j = 0, 1, \cdots, z$, then*

$$\sum_{i=0}^{|V|-2} x_i \geq \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor.$$

Moreover, if $l_1 - l_2 \geq 2$ then the inequality is strict if one of the following holds: 1) $u_{t_j-1}, u_{t_j+1} \in V_k - \{v_{0,0}\}$ for some $1 \leq k \leq n$ and $0 \leq j \leq z$; 2) v_{1,l_1-j} and $v_{1,l_1-j'}$ are consecutive for some $0 \leq j < j' \leq z$; 3) $x_i > 0$ for some $i \notin \{t_j, t_{j-1} : j = 1, 2, \cdots, z\}$.

Proof. It is trivial if $l_1 - l_2 \leq 1$, as $x_i \geq 0$ for all i. Assume $l_1 - l_2 \geq 2$ and $1 \leq t_j \leq |V| - 2$ for all $j = 0, 1, \cdots, z$. By Lemma 7, we have

$$\begin{aligned} \sum_{i=0}^{|V|-2} x_i &\geq (l_1 - l_2)(z+1) - \sum_{j=0}^{z}(2j+1) \\ &= (l_1 - l_2)(z+1) - (z+1)^2 \\ &= \lfloor \tfrac{l_1 - l_2}{2} \rfloor \lceil \tfrac{l_1 - l_2}{2} \rceil. \end{aligned}$$

The moreover part follows by Lemmas 8, 7, and the definition of x_i. ∎

Lemma 10 *Let f be a radio labeling for $G = S_{l_1,l_2,\cdots,l_n}$ with ordering of $V(G) = U(f) = (u_0, u_1, u_2, \cdots, u_{|V|-1})$. Then*

$$2\left(\sum_{i=1}^{|V|-2} L(u_i)\right) + L(u_0) + L(u_{|V|-1}) \leq \sum_{i=1}^{n} l_i(l_i+1) - 1.$$

Moreover, the equality holds if and only if $\{u_0, u_{|V|-1}\} = \{v_{0,0}, v_{t,1}\}$ for some $1 \leq t \leq n$.

Proof. In the left-side of the inequality, each vertex appears twice, except the two ends (u_0 and $u_{|V|-1}$), for which each appears once. Hence, the largest possible value is when the two ends are of the smallest levels, implying $\{u_0, u_{|V|-1}\} = \{v_{0,0}, v_{s,1}\}$ for some $1 \leq s \leq n$. ∎

Theorem 11 *Let $G = S_{l_1,l_2,l_3,\cdots,l_n}$. Then*

$$\mathrm{rn}(G) \geq \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor + 1.$$

Moreover, f is a radio labeling with span equal to this bound if and only if all the following hold: (Note, (b, c, d) are only for the case that $l_1 - l_2 \geq 2$.)

(a) $\{u_0, u_{|V|-1}\} = \{v_{0,0}, v_{s,1}\}$ *for some* s.

(b) $1 \leq t_j \leq |V| - 2$, *for all* $0 \leq j \leq z$.

(c) $x_{t_j - 1} + x_{t_j} = l_1 - l_2 - (2j+1)$, *for all* $0 \leq j \leq z$.

(d) *For any* $0 \le j \le z$, u_{t_j-1} *and* u_{t_j+1} *belong to different legs, unless one of them is* $V_{0,0}$.

(e) *If* $l_1 - l_2 \le 1$, *then* $x_i = 0$ *for all* $0 \le i \le |V| - 2$;
if $l_1 - l_2 \ge 2$, *then* $x_i = 0$ *for all* $i \notin \{t_j, t_j - 1 : j = 0, 1, \cdots, z\}$.

Proof. Let f be a radio labeling for $G = S_{l_1,l_2,\cdots,l_n}$. Consider the case that $l_1 - l_2 \le 1$, or $l_1 - l_2 \ge 2$ and $1 \le t_j \le |V| - 2$ for any $v_{1,l_1-j} = u_{t_j}$, $0 \le j \le z$. By the definition of x_i and Lemmas 9 and 10, we get

$$\begin{aligned} f(u_{|V|-1}) &= (l_1 + l_2 + 1)(|V| - 1) - 2\sum_{i=1}^{|V|-2} L(u_i) - L(u_0) - L(u_{|V|-1}) + \sum_{i=0}^{|V|-2} x_i \\ &\ge (l_1 + l_2 + 1)\sum_{i=1}^{n} l_i - \sum_{i=1}^{n} l_i(l_i + 1) + 1 + \lceil \tfrac{l_1-l_2}{2} \rceil \lfloor \tfrac{l_1-l_2}{2} \rfloor \\ &= \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \lceil \tfrac{l_1-l_2}{2} \rceil \lfloor \tfrac{l_1-l_2}{2} \rfloor + 1. \end{aligned}$$

Moreover, the second equality in the above holds if and only if (a, c, d, e) are true.

It remains to show that if (b) fails, then the span of f is greater than the desired bound. Assume $t_j = 0$ for some $j = 0, 1, 2, \cdots, z$ and $t_{j'} \le |V| - 2$ for all $j' = 0, 1, \cdots, z$. Similar to the proof of Lemma 9, we get

$$\sum_{i=0}^{|V|-2} x_i \ge \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor - (l_1 - l_2 - 2j - 1).$$

The result then follows by a similar calculation to the previous paragraph. By the same method, one can show that if $t_j = 0$ and $t_{j'} = |V| - 1$ for some $0 \le j, j' \le z$, then the span of f is also greater than the desired bound. ∎

In the next two results, we characterize the spiders whose radio numbers achieve the bounds in Theorem 11 and Corollary 5, respectively.

Theorem 12 *Let* $G = S_{l_1,l_2,\cdots,l_n}$ *be a spider with* $n \ge 3$. *Then*

$$\mathrm{rn}(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + \lceil \frac{l_1 - l_2}{2} \rceil \lfloor \frac{l_1 - l_2}{2} \rfloor + 1$$

if and only if $\overline{l_1} \ge \frac{l_1+l_2-1}{2}$.

Proof. Consider cases.

Case $l_1 - l_2 \leq 1$ Then $\overline{l_1} \geq \frac{l_1+l_2-1}{2}$. It suffices to give radio labelings with the desired spans.

In all the following cases, we use a diagram to describe a labeling f. First, we fix the ordering of the vertices $V = U(f) = (u_0, u_1, \cdots, u_{|V|-1})$. Note that if $p > l_i$, then $v_{i,p}$ does not exist. For all the diagrams given, when encountering such a "non-existing vertex," we simply skip it and move on to the next vertex. Secondly, we put a sign $\xrightarrow{\ell}$ between two consecutive vertices u_i and u_{i+1} to indicate that $x_i = \ell$. In the case $x_i = 0$, we just put "$\rightarrow$" between u_i and u_{i+1}. With the assumption that $f(u_0) = 0$, it is easy to see that the labeling f is well-defined.

It is a routine to check that each given labeling is a radio labeling (by Lemma 6 (1) (2)) with the desired span (by Theorem 11). We shall sketch the proof only for the first one below and leave the details of others to the reader. To make the labeling more visual, an example is provided.

If $l_1 = l_2$, then f is defined by: (See Figure 2 for an example.)

$$\begin{array}{lllll}
v_{0,0} & \rightarrow v_{1,l_1} & \rightarrow v_{2,1} & \rightarrow v_{3,l_1} & \rightarrow v_{4,l_1}\cdots & \rightarrow v_{n,l_1} \\
& \rightarrow v_{1,l_1-1} & \rightarrow v_{2,2} & \rightarrow v_{3,l_1-1} & \rightarrow v_{4,l_1-1}\cdots & \rightarrow v_{n,l_1-1} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& \rightarrow v_{1,1} & \rightarrow v_{2,l_1} & \rightarrow v_{3,1} & \rightarrow v_{4,1}\cdots & \rightarrow v_{n,1}.
\end{array}$$

It is easy to see that the span of f equals to the desired value, since $x_i = 0$ for all i, and $u_0, u_{|V|-1} = \{v_{0,0}, v_{n,1}\}$. To verify that f is a radio labeling, it suffices to show that f satisfies Lemma 6 (1)(2). Because $2L(u_i) < 2l_1 + 1 = l_1 + l_2 + 1$ for any $0 \leq i \leq |V| - 1$, so (1) holds. To show (2), consider a set of consecutive vertices $\{u_i, u_{i+1}, \cdots, u_j\}$ with $u_i, u_j \in V_k$ for some k. By the definition of f in the above, if $k \geq 3$, then $j - i \geq 3$, and there exists some $i+1 \leq q \leq j-1$, such that $2\left(L(u_q) + \min\{L(u_i), L(u_j)\}\right) \leq l_1 + l_2 + 2$. Combining this with the fact that $2L(u_s) < l_1 + l_2 + 1$ for every s, (2) is true. If $k \leq 2$, then there exists some $i + 1 \leq q \leq j - 1$ such that $2\left(L(u_q) + \min\{L(u_i), L(u_j)\}\right) \leq l_1 + l_2 + 1$, so (2) holds.

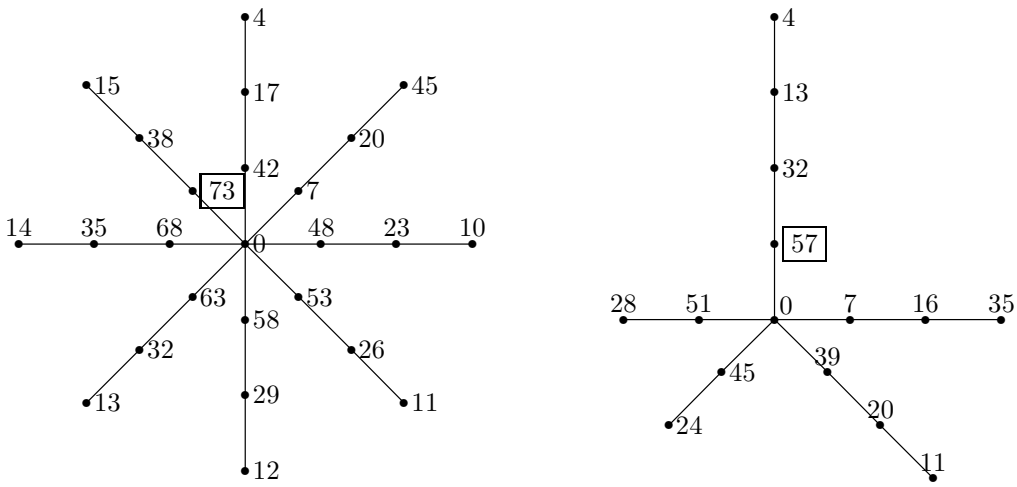


Figure 2: Optimal radio labelings for $S_{3,3,3,3,3,3,3,3}$ and $S_{4,3,3,2,2}$.

If $l_1 - l_2 = 1$, the labeling is given by: (See Figure 2 for an example.)

$$
\begin{array}{llllllll}
v_{0,0} & \to v_{1,l_1} & \to v_{2,1} & \to v_{3,l_2} & \to v_{4,l_2} & \to \cdots & \to v_{n,l_2} \\
& \to v_{1,l_1-1} & \to v_{2,2} & \to v_{3,l_2-1} & \to v_{4,l_2-1} & \to \cdots & \to v_{n,l_2-1} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\
& \to v_{1,2} & \to v_{2,l_2} & \to v_{3,1} & \to v_{4,1} & \to \cdots & \to v_{n,1} \\
& \to v_{1,1}. & & & & &
\end{array}
$$

Case $l_1 - l_2 \geq 2$ First, we prove that if the bound is achieved, then $\overline{l_1} \geq \frac{l_1+l_2-1}{2}$. Note, it is trivial if $l_1 - l_2 \leq 3$, as $l_3 \geq 1$.

Assume $l_1 - l_2 \geq 4$. Let f be a radio labeling for G with span equal to the desired bound. We adopt the same notations used in the proof of Theorem 11, let $z = \lfloor \frac{l_1-l_2-2}{2} \rfloor$, and let $v_{1,l_1-j} = u_{t_j}$ for $j = 0, 1, \cdots, z$. Then, (a) - (e) in Theorem 11 hold.

Claim. For any $\lfloor \frac{l_1-l_2+1}{2} \rfloor \leq i \leq \lceil \frac{l_1+l_2+1}{2} \rceil$, $v_{1,i}$ is not consecutive to any vertex in $V_1 - \{v_{0,0}\}$.

Proof. Let $v_{1,i} = u_q$ for some q. By Theorem 11 (a), $1 \leq q \leq |V| - 2$, as $i \geq \lfloor \frac{l_1 - l_2 + 1}{2} \rfloor \geq 2$. Note, $i \leq l_1 - z$. Assume $i = l_1 - z$. Then $l_1 - l_2$ is even and $q = t_z$. By Theorem 11 (c), $x_{q-1} + x_q = l_1 - l_2 = 1$. By Lemma 6 (2), $v_{1,i}$ can not be consecutive to any vertex in $V_1 - \{v_{0,0}\}$.

Assume $i < l_1 - z$. By Lemma 6 (2) and Theorem 11 (e), it is enough to show that $q \neq t_j - 1$, for all $j = 0, 1, 2, \cdots, z$. Suppose to the contrary, $q = t_j - 1$ for some $0 \leq j \leq z$. Since $l_1 - z > i$, by Lemma 6 (2), $x_{t_j - 1} \geq 2L(u_q) = 2i \geq l_1 - l_2$, contradicting Theorem 11 (c). ∎

Suppose $l_1 - l_2$ is odd. Let $A = \{v_{1,i} : i = 1, 2, \cdots, z+1\}$, $B = \{v_{1,l_1-j} : j = 0, 1, \cdots, z\}$, and $C = V_1 - (A \cup B \cup \{v_{0,0}\})$. By Theorem 11 (d) and the Claim, any vertex in A, B, and C, respectively, can only possibly be consecutive to vertices in $V(G) - (C \cup A)$, $V(G) - (C \cup B)$, and $(V(G) - V_1) \cup \{v_{0,0}\}$. As $|A| = |B| = z+1$, we conclude that $|V(G) - V_1| \geq z + |C| = \frac{l_1 + l_2 - 1}{2}$, implying $\overline{l_1} \geq \frac{l_1 + l_2 - 1}{2}$. Similarly, one can show that the result holds when $l_1 - l_2$ is even. We leave this to the reader.

It remains to give a radio labeling with span equal to the desired bound. We consider cases separately. If $l_1 - l_2 = 2$, f is defined by: (See Figure 3 as an example.)

$$
\begin{array}{llllllll}
v_{0,0} & \to v_{1,l_1} & \xrightarrow{1} v_{2,1} & & & & & \\
 & \to v_{1,l_1-1} & & \to v_{3,1} & \to v_{4,1} & \to v_{5,1} & \cdots & \to v_{n,1} \\
 & \to v_{1,l_1-2} & \to v_{2,2} & \to v_{3,2} & \to v_{4,2} & \to v_{5,2} & \cdots & \to v_{n,2} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \to v_{1,2} & \to v_{2,l_2} & \to v_{3,l_2} & \to v_{4,l_2} & \to v_{5,l_2} & \cdots & \to v_{n,l_2} \\
 & \to v_{1,1}. & & & & & &
\end{array}
$$

If $l_1 - l_2 \geq 3$, we consider two sub-cases. Let A be the set of vertices, $A = V(G) - (V_1 \cup V_2)$. We line up the vertices in A by:

$$A = (v_{3,1}, v_{4,1}, \cdots, v_{n,1}, v_{3,2}, v_{4,2}, \cdots, v_{n,2}, \cdots, v_{n,l_3}).$$

By assumption and by considering the parity of $l_1 - l_2$, we get $|A| \geq z + 1$.

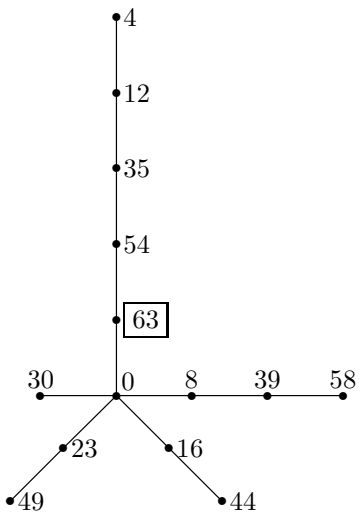


Figure 3: An optimal radio labeling for $S_{5,3,2,2,1}$.

Let $A[z+1]$ be the set of the first $z+1$ vertices in A, and let

$$A_i = \{v : L(v) = i\} \cap (A - A[z+1]).$$

We denote $A(z+1)$ the first unlabeled yet vertex in the line up of $A[z+1]$. When we encounter A_i in the diagrams below, we color *all* the vertices in A_i one by one (in any order) with $x_q = 0$ if $u_{q-1}, u_q \in A_i$. If $A_i = \emptyset$, we skip it.

For both sub-cases, see Figure 4 for examples.

Sub-case: $l_1 - l_2$ is even. Then $z = \frac{l_1 - l_2 - 2}{2}$. The labeling is defined as:

$$
\begin{array}{lll}
v_{0,0} & \to v_{1,l_1-z} & \stackrel{1}{\to} A(z+1) \\
 & \to v_{1,l_1-z-1} & \to v_{2,1} \quad \to A_{l_2} \\
 & \to v_{1,l_1-z-2} & \to v_{2,2} \quad \to A_{l_2-1} \\
 & \vdots & \vdots \quad \vdots \\
 & \to v_{1,z+2} & \to v_{2,l_2} \quad \to A_1 \\
 & \to v_{1,z+1} & \to A(z+1) \\
 & \to v_{1,l_1} & \stackrel{2z+1}{\longrightarrow} v_{1,z} \quad \to A(z+1) \\
 & \to v_{1,l_1-1} & \stackrel{2z-1}{\longrightarrow} v_{1,z-1} \quad \to A(z+1) \\
 & \vdots & \vdots \quad \vdots \\
 & \to v_{1,l_1-z+2} & \stackrel{5}{\to} v_{1,2} \quad \to A(z+1) \\
 & \to v_{1,l_1-z+1} & \stackrel{3}{\to} v_{1,1}.
\end{array}
$$

Sub-case: $l_1 - l_2$ is odd. Then $z = \frac{l_1-l_2-3}{2}$. The labeling is defined by:

$$
\begin{array}{lll}
v_{0,0} & \to v_{1,l_1-z-1} & \to v_{2,1} \quad \to A_{l_2} \\
 & \to v_{1,l_1-z-2} & \to v_{2,2} \quad \to A_{l_2-1} \\
 & \vdots & \vdots \quad \vdots \\
 & \to v_{1,z+3} & \to v_{2,l_2} \quad \to A_1 \\
 & \to v_{1,z+2} & \to A(z+1) \\
 & \to v_{1,l_1} & \stackrel{2(z+1)}{\longrightarrow} v_{1,z+1} \quad \to A(z+1) \\
 & \to v_{1,l_1-1} & \stackrel{2z}{\longrightarrow} v_{1,z} \quad \to A(z+1) \\
 & \vdots & \vdots \quad \vdots \\
 & \to v_{1,l_1-(z-1)} & \stackrel{4}{\to} v_{1,2} \quad \to A(z+1) \\
 & \to v_{1,l_1-z} & \stackrel{2}{\to} v_{1,1}.
\end{array}
$$

∎

Now we turn our attention to the spiders achieving the bound in Corollary 5. Notice, the case $l_1 - l_2 \leq 1$ has been determined in Theorem 12. Hence, we assume $l_1 - l_2 \geq 2$ and $n \geq 3$.

Theorem 13 *Let $G = S_{l_1,l_2,\cdots,l_n}$ be a spider with $l_1 - l_2 \geq 2$ and $n \geq 3$. Then the equality in Corollary 5 holds if and only if $l_3 = 1$, $n = 3$, and $l_1 - l_2$ is odd.*

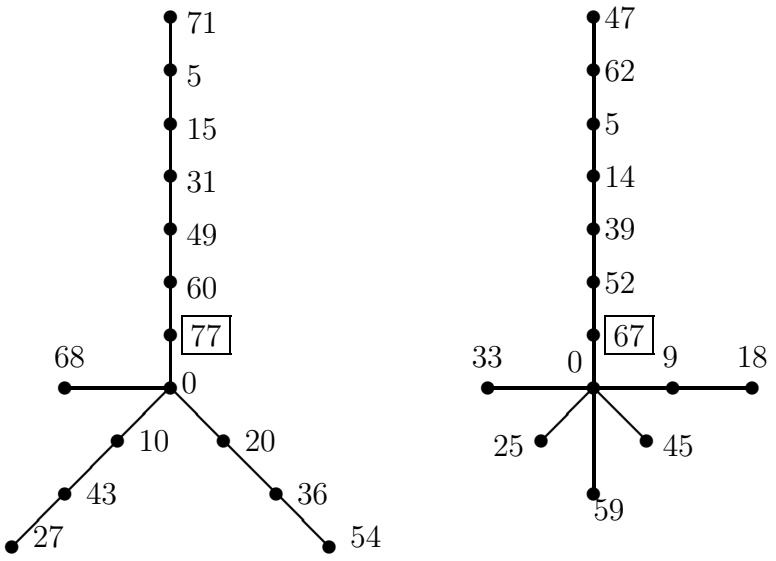


Figure 4: Optimal radio labelings for $S_{7,3,3,1}$ and $S_{7,2,1,1,1,1}$.

Proof. Assume $l_1 - l_2 \geq 2$, $n \geq 3$, and the equality in Corollary 5 holds. Let f be an optimal radio labeling with ordering $U(F) = (u_0, u_1, u_2, \cdots, u_{m-1})$, where $m = l_1 + l_2 + \cdots + l_n + 1$. If $l_1 \leq \overline{l_1} + 1$, then $\mathrm{rn}(G) = \sum_{i=1}^{n} l_i(l_1 + l_2 - l_i) + 1$, contradicting Theorem 11 (as $l_1 - l_2 \geq 2$).

Hence, $l_1 > \overline{l_1} + 1$. By Lemmas 1 and 2, and Theorem 3, we assume, by symmetry, that a weight center is $w^* = v_{1,k} = u_0$, where $k = l_1 - \lfloor m/2 \rfloor$. There are exactly two branches for the root w^*. By Theorems 3, 4, and Corollary 5, f satisfies (1 - 3) in Theorem 3.

Claim. If $u_q = v_{1,l_1}$, then $1 \leq q \leq m-2$ and

$$\min\{d(u_{q-1}, u_q), d(u_q, u_{q+1})\} \leq (l_1 + l_2 + 1)/2.$$

Proof) Assume $u_q = v_{1,l_1}$. By Theorem 3 (2), $1 \leq q \leq m-2$ (since $l_1 \geq 2$). Assume $d(u_{q-1}, u_q) \geq d(u_q, u_{q+1})$ (the other case is similar). Then $d(u_{q-1}, u_q) + d(u_q, u_{q+1}) - d(u_{q-1}, u_{q+1}) \geq 2\min\{l_1, d(u_{q+1}, u_q)\}$. By Theorem (3) and definition, we obtain $2(d+1) - d(u_{q-1}, u_q) - d(u_q, u_{q+1}) = f(u_{q+1}) - f(u_{q-1}) \geq d + 1 - d(u_{q-1}, u_{q+1})$. So the result follows as $d = l_1 + l_2 + 1$. ∎

Assume $l_3 + l_4 + \cdots + l_n \geq 2$. Then $d(v_{1,l_1}, v_{1,k}) = l_1 - k \geq (l_1 + l_2 + 2)/2$. This implies that $d(v_{1,l_1}, v) > (l_1 + l_2 + 1)/2$, for any vertex v on the branch opposite to v_{1,l_1}, contradicting Theorem 3 (1) and the Claim. Therefore, $l_3 + \cdots + l_n \leq 1$. Since $n \geq 3$, it follows that $l_3 = 1$ and $n = 3$. Assume $l_1 - l_2$ is even. Then $m = l_1 + l_2 + 2$ is even, and $l_1 - k = m/2$. So the distance between v_{1,l_1} and any other vertex in the opposite branch is at least $m/2 > (l_1 + l_2 + 1)/2$, contradicting the Claim.

It remains to give a radio labeling f satisfying (1 - 3) in Theorem 3, for $l_3 = 1$, $n = 3$, and $l_1 - l_2 \geq 3$ is odd. By Lemma 1, $w^* = v_{1,k}$ is the weight center, where $k = (l_1 - l_2 - 1)/2$. Define the ordering $U(f)$ by the following three steps, and for each i, let $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$:

1) $u_0 = v_{1,k}$, $u_1 = v_{1,l_1}$, $u_2 = v_{1,k-2}$ (or $u_2 = v_{2,1}$ if $k = 1$);

2) move back and forth on the path $V_1 \cup V_2$, about the weight center $v_{1,k}$, with distances alternating between $(l_1 + l_2 + 1)/2$ and $(l_1 + l_2 + 3)/2$, until we reach the vertex $v_{1,k+1}$. That is, u_3 is a vertex with distance $(l_1 + l_2 + 1)/2$ from u_2 (indeed $u_3 = v_{1,l_1-2}$), and u_4 has distance $(l_1 + l_2 + 3)/2$ away from u_3, etc;

3) $u_{m-3} = v_{3,1}$, $u_{m-2} = v_{1,l_1-1}$, and $u_{m-1} = v_{1,k-1}$.

It is straightforward to check that f is a radio labeling satisfying (1 - 3) in Theorem 3. ∎

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