Edge-Nim on a Tetrahedron

Recreational Mathematics
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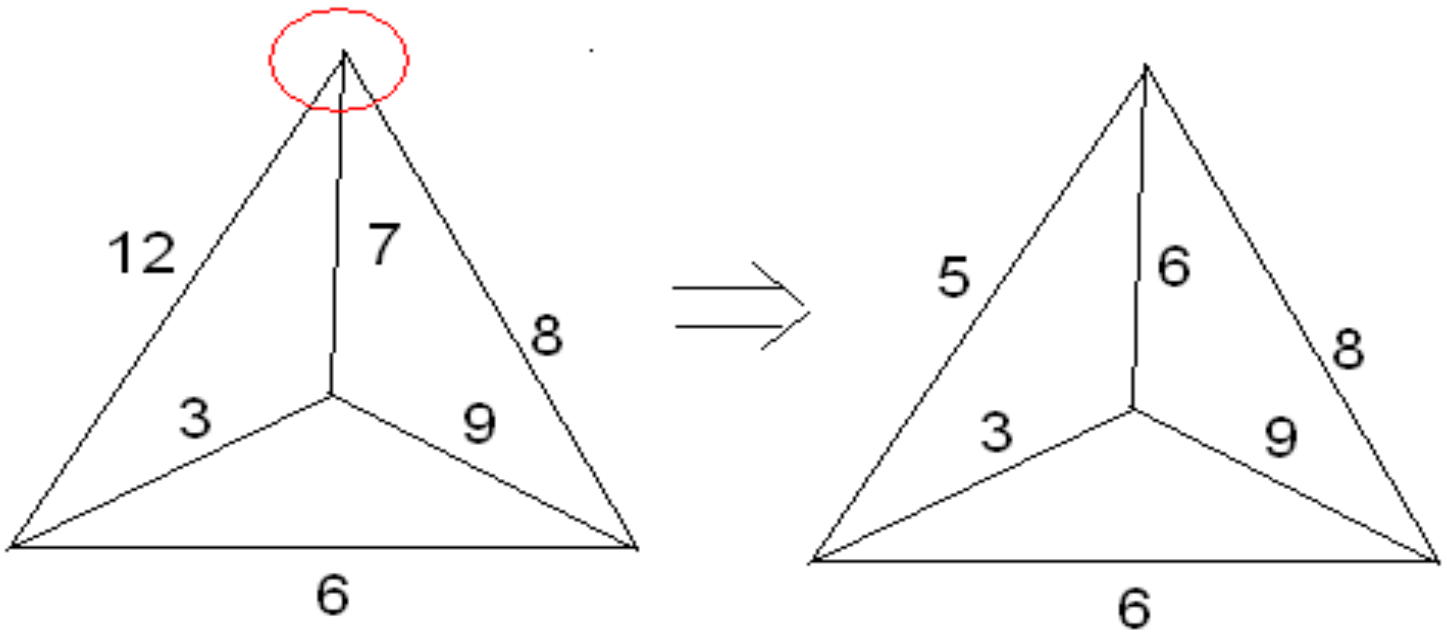
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Definition of Edge-Nim

- Tokens are placed on the **edges** of a graph
- Two players alternate
- A move consists of
  1. ...choosing a vertex;
  2. ...removing **at least** one token from **one or more edges** incident to the chosen vertex

As usual in combinatorial games, the first player who cannot move loses. Or, equivalently, the player who removes the last token wins.

Example of a Move:



History of the Game

- At the 2013 Boca Raton conference, Breeann gave a talk on this game
- She had used it to introduce prospective teachers to research via graph theory problems
- She gave results on C_3 and C_4, and indicated that **Edge-Nim on a Tetrahedron** was open
- **Edge-Nim** on C_3 and C_4 are equivalent to **Circular Nim** $C(3,2)$ and $C(4,2)$

We are interested in a winning strategy. We will determine the **losing positions (P-positions)**, those from which a player will be guaranteed to lose.

This needs the **digital sum of integers** (denoted by $\oplus$)

To compute, say $12 \oplus 17 \oplus 5$:

1. First express them in base 2:

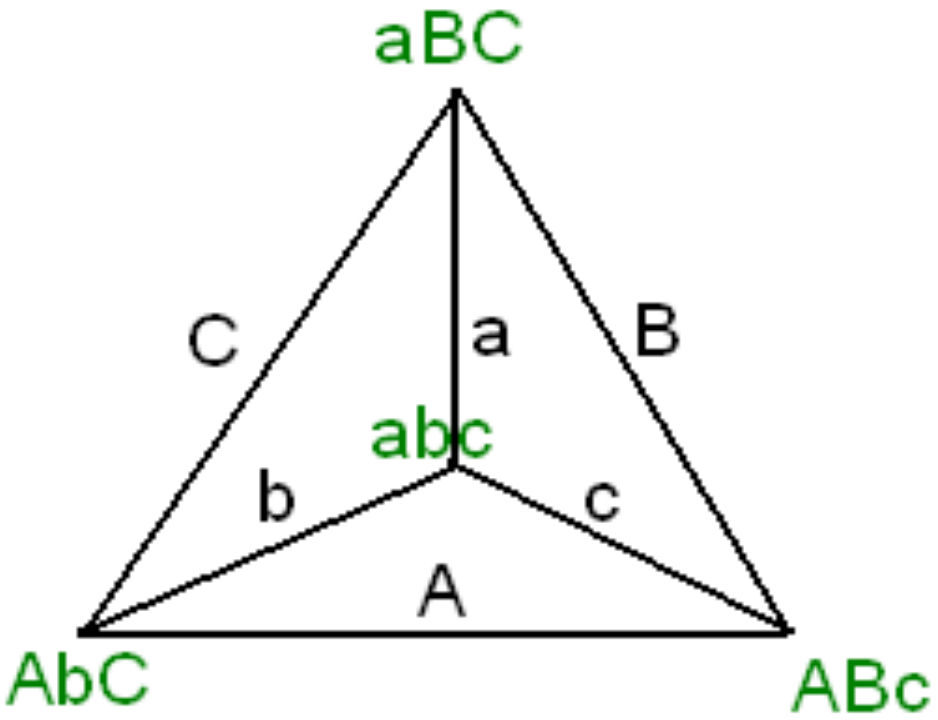
   $12 = 01100_2$ $17 = 10001_2$ $5 = 00101_2$

2. Add them without «carrying over»:

   $01100_2 \oplus 10001_2 \oplus 00101_2 = 11000_2$

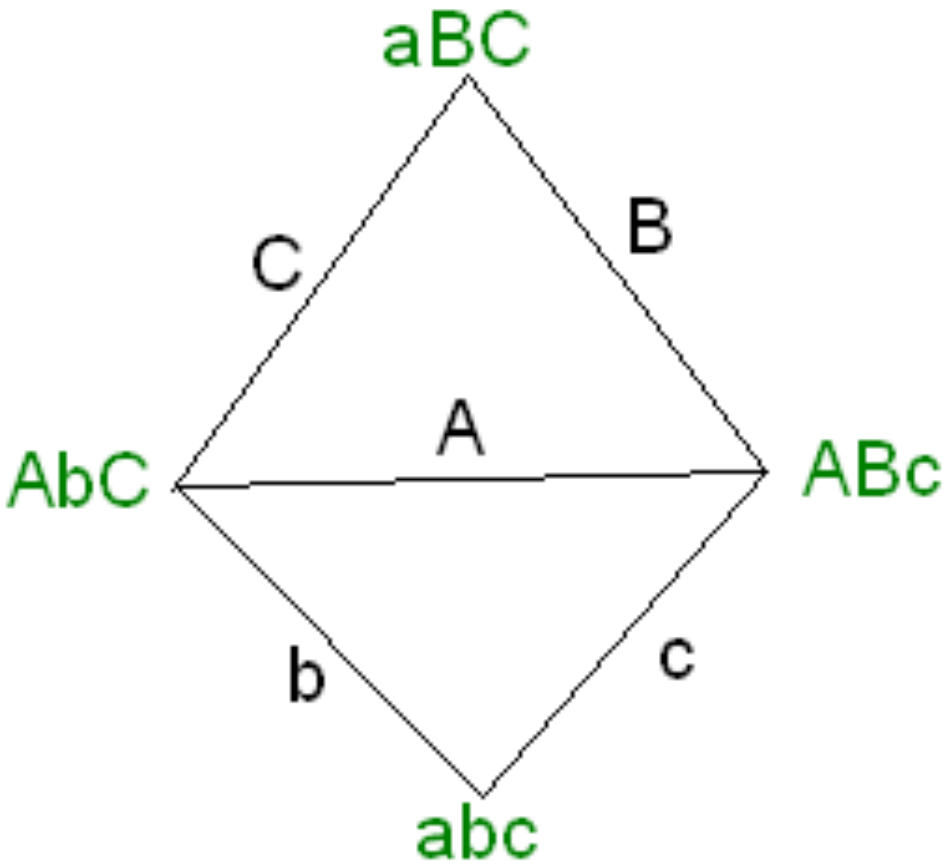
3. Rewrite the number in base 10: **24**.

Here is how one will identify the edges (in black) and the vertices (in green) of our tetrahedron:



Vertices labeled in alphabetical order.

Special Case: If $a = 0$, it is like if that edge was not there anymore, as no move can be made on it. So flip down the edges b and c for convenience.

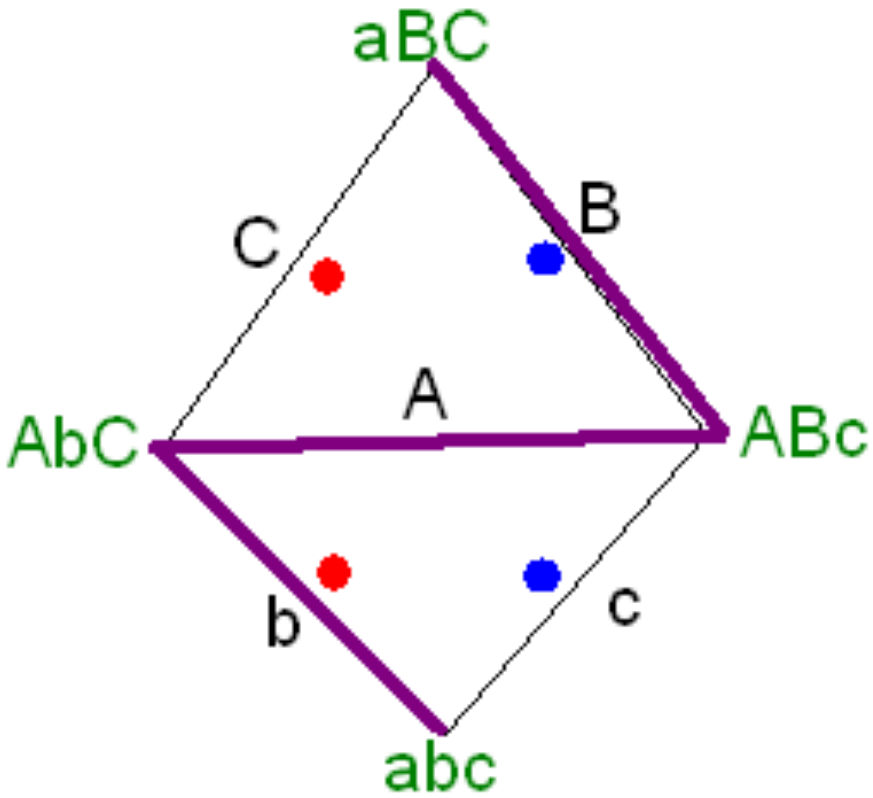


Partial result for the game when one of the edges is empty

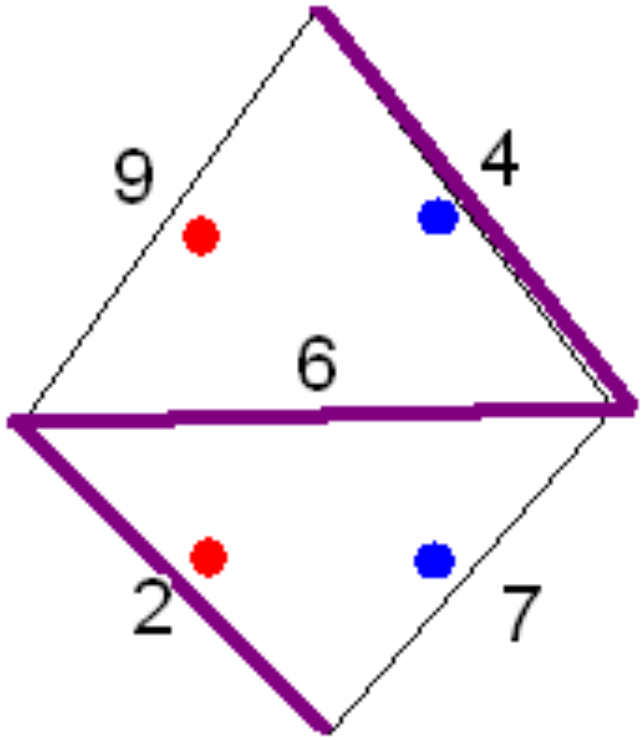
The set *L* of **losing positions** for the case *a* = 0 consists of all tetrahedra for which

1. b + C = B + c
2. b ≤ c
3. A ⊕ b ⊕ B = 0

...so, red edges add to the same as the blue edges, and purple edges have digital sum 0



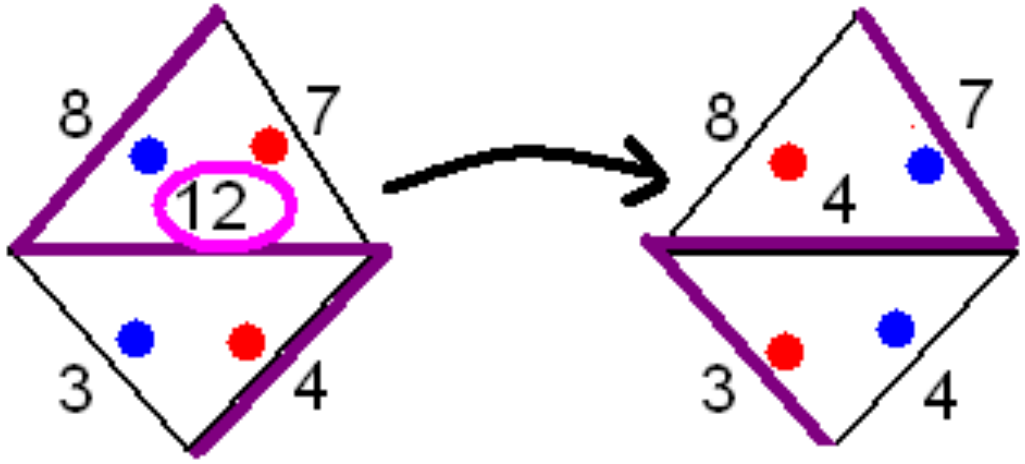
Example: This position belongs to *L*:



$9+2 = 4+7=11$

$4 = 100_2$
$6 = 110_2$
$2 = 010_2$
000_2

Here we show that Condition 2 ($b \le c$) is important and not just an expression of convenience to draw the pictures. Indeed, if one drops this condition (which is equivalent to flipping the path along the (missing vertical) edge a:



...and one would have a move from L to L, which is forbidden. So Condition 2 is important!

Proof Method:

Once one has a guess S for the set of losing positions L, one must show that:

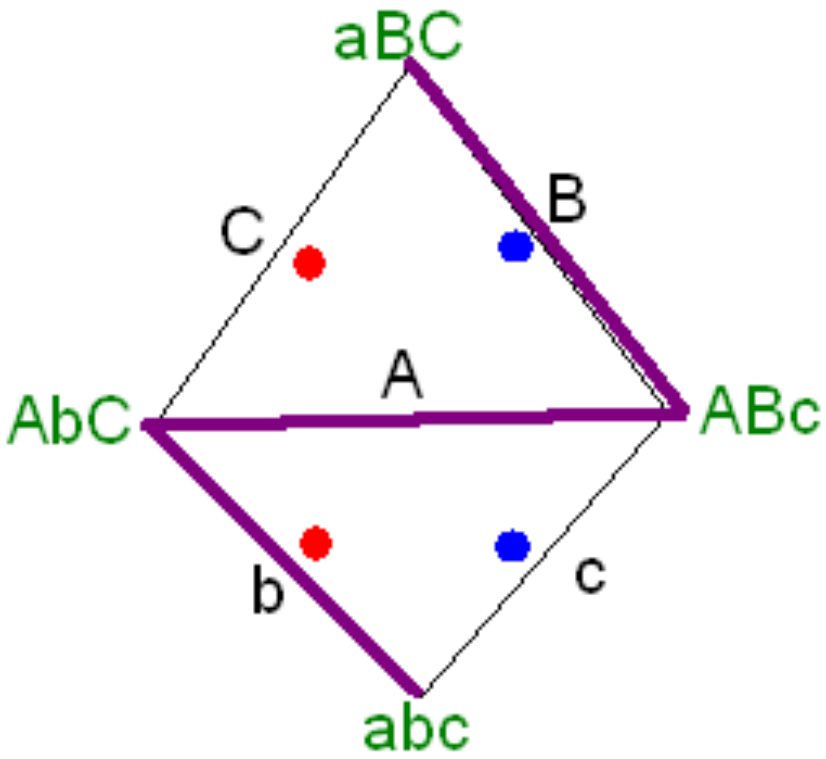
1. From a position in S, one cannot play to another position in S.

2. From a position outside S, one can always play to a position in S.

Then conclude $S = L$, the set of losing positions, and the game is solved...

Part 1:

- Play on either AbC or ABc can only be play on A (otherwise equality of sums is destroyed.
- Play on either aBC or abc must reduce both edges by the same amount to maintain equality of sums.
- In both cases, exactly one of the edges on the "digital" path is modified, which results in making the digital sum non-zero



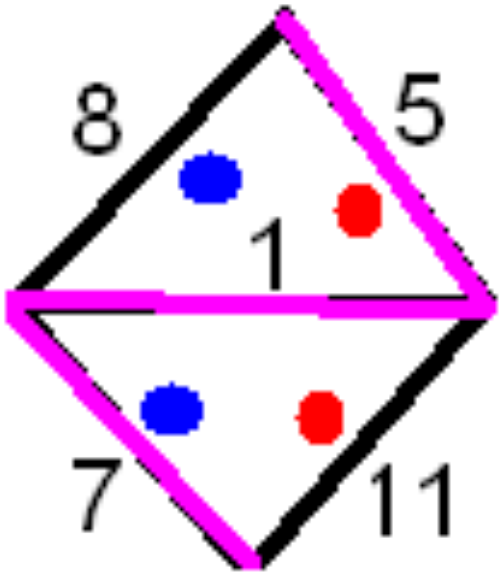
⇒ move from position in S is to position not in S.

Part 2: The proof is tedious. One needs to

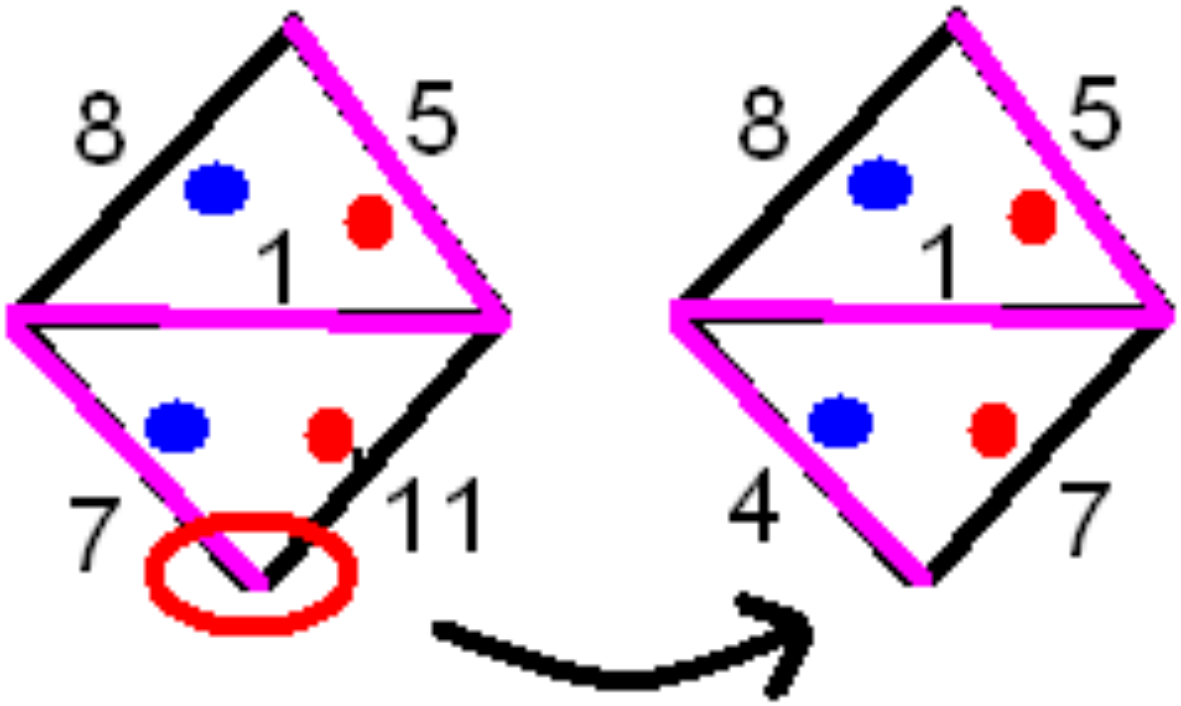
- decompose the set S into subsets with conditions like $b \leq c$, $A \leq b \oplus B$, etc. and make sure that all positions are covered
- determine how to play in each case...no obvious insights.

Let's have some fun:

How should one play here?



Answer:



Thanks!