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# *Mathematical Platonism*[1]

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## *From the Editors*

*"Platonism" as a philosophy of mathematics refers back to Plato's dialogues on the Forms, which have been represented as existing in some eternal, unchanging, non-physical realm. Platonism in mathematics locates mathematical objects there. Many mathematicians believe that platonism, as a philosophy of mathematics has been discredited, in part due to the contradictions of naive set theory, in part because of the question of how we physical beings could contact such a realm. However, in fact platonism remains the default philosophy of mathematics among philosophers, one that few are willing to defend in the strong form attributed to Gödel, but which will not be replaced until a satisfactory alternative has been found.*

*This chapter summarizes over forty years of such discussion, between those defending some version of platonism (called platonists, or realists) and those opposing platonism, usually called nominalists. Because it is summarizing discussion that has developed in several hundred articles and dozens of books, this chapter is not one to read casually. However, this chapter is very clearly and comprehensively structured, so that those who take the effort it to read it will be rewarded with a thorough survey of the many different schools of thought that have developed among philosophers during this period. Thus this chapter provides an excellent introduction for anyone who would like to be able to start reading original work by philosophers of mathematics.*

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[1] Many of the ideas in this essay, and in some cases the wording of the ideas, come from my [1998a] and my [2004].

*of Mathematics,"* Philosophical Studies *(1996), "Towards a Nominalization of Quantum Mechanics,"* Mind *(1996), "A Platonist Epistemology,"* Synthese *(1995), and has an article soon to be published, "Realism and Anti-Realism in Mathematics," in a forthcoming book,* Handbook of the Philosophy of Mathematics, *edited by Andrew Irvine. He has published one book,* Platonism and Anti-Platonism in Mathematics *(1998), which does in a much more extensive way what this chapter does: it explains the various platonist (realist) and anti-platonist (nominalist) approaches to the philosophy of mathematics, and attempts to show that there is only one plausible version of each, and that these two versions are effectively indistinguishable.*

## 1 Introduction

Philosophers of mathematics are interested in the question of what our mathematical sentences and theories are *about*. These sentences and theories *seem* to be making straightforward claims about certain objects. Consider, for instance, the sentence '3 is prime.' This sentence seems to be a simple subject-predicate sentence of the form 'The object *a* has the property *F*'—like, for instance, the sentence 'The moon is round.' This latter sentence seems to make a straightforward claim about the moon. Likewise, the sentence '3 is prime' seems to make a straightforward claim about the number 3. But this is where philosophers get puzzled. For it's not clear what the number 3 is supposed to *be*. What kind of thing is a *number*? Some philosophers (anti-realists, or nominalists) have responded here with disbelief: according to them, there are simply no such things as numbers. Others (the realists) think that there *are* such things as numbers (as well as other mathematical objects, such as sets). But among the realists, there are several different views of what *kind of thing* a number is. Some people have thought that numbers are mental objects (something like ideas in our heads). Others have claimed that numbers exist outside of our heads, as features of the physical world. The historically most popular view, however, called ***platonism***, is the view that numbers are ***abstract objects***. An abstract object is an object that exists outside of space and time. These objects (if there really are such things) are wholly non-physical, non-mental, and causally inert. In other words, they do not enter into causal relations with other objects. (The fact that abstract objects are non-causal in this way follows from the fact that they're nonspatiotemporal. Since they're not extended in space, and since they aren't made of physical *stuff*, they cannot enter into cause-and-effect relationships. They cannot cause other objects to *move* in the way that, say, a cue ball can.) So according to platonists, 3 is a real and objective thing that, like the moon, exists independently of us and our thinking (that is, it's not just an idea in our heads). But 3 is also *different* from the moon, according to platonists, because it's not a *physical* thing. That is, numbers exist (really and objectively and independently of us and our thoughts), but they do not exist in space and time.

Given these remarks, it might seem that a "philosophy of mathematics" is essentially an ***ontological*** theory. (An ontological theory is a theory about what sorts of things really exist. Thus, for instance, the claim that there are unicorns is a false ontological theory, and the claim that there are tigers is a true ontological theory.) Now, there is often (though not always) an ontological *component* to a philosophy of mathematics; but if you want to understand what philosophers of mathematics are really up to, it is important to realize that the *first* thing they want to do (and sometimes the only thing) is to construct a ***semantic*** theory. A semantic theory is a presumably empirical theory about what certain expressions mean (or refer to) in ordinary discourse. So, for instance, the claim that the term 'George W. Bush' denotes (in English) the Empire State Building is a false semantic theory, and the claim that 'George W. Bush' denotes (in English) the forty-third president of the United States is a true semantic theory. A philosophy of mathematics involves a semantic theory because it tells us how to interpret the sentences of ordinary mathematical discourse. It tells us what sorts of objects, if any, terms like '3' are supposed to refer to. For instance, platonism tells us that numerals like '3' are supposed to refer to abstract objects. Another theory (psychologism) tells us that numerals are supposed to refer to mental objects.



*Some* philosophical views about mathematics also have ontological components. For instance, platonism tells us not just that numerals like '3' are supposed to refer to abstract objects, but that there really do exist such things. But other philosophical views don't have any ontological components. For instance, some views hold that numerals aren't supposed to refer to objects at all, and so, on views like this, ontological questions (about what kinds of objects really exist) are simply irrelevant to the philosophy of mathematics. And finally, there are other philosophical views that contain *entirely uncontroversial* ontological components. For instance, the psychologistic view that says that numerals like '3' refer to ideas in our heads does contain an ontological thesis (namely, that we really do have ideas of numbers in our heads). But this ontological thesis is not very controversial: of course we have such ideas. In contrast to this, however, any theory that could accurately be called a *philosophy of mathematics* is going to contain a (controversial) semantic component.

Given this, the relationship between a mathematician and a philosopher of mathematics is analogous to the relationship between a native speaker of French and a certain sort of *linguist*—in particular, a grammarian of French whose native tongue is English but who has learned a good deal of French in order to construct a grammar for that language. There is an obvious sense in which the native speaker of French knows her language better than the linguist does. But the linguist has been trained to construct syntactic theories, and most native speakers of French have not. Thus, while the linguist has to respect the linguistic intuitions of native speakers, he cannot very well ask them what the right theory is. Likewise, while it is obvious that mathematicians know mathematics (and the language of mathematics) better than philosophers do, most of them have not been trained to construct semantic theories in the way that philosophers have. So while philosophers of mathematics have to respect the intuitions of mathematicians, they cannot very well ask them what the right theory is.

In this essay, I will provide an overview of the most important views and arguments in the philosophy of mathematics, and at the same time, I will provide some (brief and incomplete) arguments for what I think is the *right* view. In section 2, I will provide a more precise statement of the platonistic view of mathematics, and then I will formulate the central argument in favor of this view. In the process of running through this argument and the various possible responses to it, I will also provide a description and critique of the various *alternatives* to platonism. (These two tasks naturally go together, because the main argument for platonism is centrally concerned with showing that none of the alternatives to platonism is plausible. And this should not be surprising. For while platonism seems to provide a very satisfying account of our mathematical theories, it can be pretty hard to swallow from an ontological point of view. After all, the platonist's ontological

thesis—that there are wholly non-physical and non-mental objects that exist outside of space and time—seems rather weird and mysterious. One might simply find it hard to believe that there really exist such things as abstract objects. But the problem is that it's very hard to find a plausible alternative to platonism. And this is the basis of the central argument in favor of platonism. The idea is to try to show that none of the alternatives is capable of providing a satisfactory account of mathematical theory and mathematical practice.) After running through the central argument for platonism in section 2, I will move on in section 3 to a discussion of what is widely thought to be the best argument *against* platonism. My own view is that in the end, neither of these two arguments succeeds, and in discussing these arguments, I will explain why I think they fail. But since this essay is primarily a survey piece, my remarks will have to be rather sketchy and incomplete. (I will, however, refer the reader to other works that fill in the details.) Finally, in section 4, I will provide a few concluding remarks about what I think we ought to say about the question of whether platonism is true and, more generally, about the philosophy of mathematics.

## 2 The Fregean Argument for Mathematical Platonism

(and a Taxonomy of the Alternatives to Platonism)

### 2.1 The Argument

Mathematical platonism, formally defined, is the view that (a) there exist abstract mathematical objects—objects that are non-spatiotemporal and wholly non-physical and non-mental—and (b) our mathematical theories are true descriptions of such objects. This view has been endorsed by a number of different people, including Plato, Frege, Gödel, and in some of his writings, Quine.[2]

The central argument for platonism is due mainly to Frege ([1884] and [1893–1903]), although I will present it somewhat differently than he did. The argument can be put like this:

(1) Our mathematical theories are extremely useful in empirical science—indeed, they seem to be indispensable to our empirical theories—and the only way to account for this is to admit that our mathematical theories are true. Therefore,
(2) The sentences of our mathematical theories—sentences like '3 is prime'—are true. Moreover, it seems that
(3) Sentences like '3 is prime' should be read at face value. (Philosophers would put this by saying that the logical form of '3 is prime' is '*a* is *F*,' where '*a*' is a constant and '*F*' is a predicate. Thus, the claim here is that '3 is prime' has the same logical form as, e.g., 'Mars is red.' Both sentences just make straightforward claims about the nature of certain objects. The one makes a claim about the nature of Mars, and the other makes a claim about the nature of the number 3.) But
(4) If we allow that sentences like '3 is prime' are true, and if moreover we allow that they should be read at face value, then we are committed to believing in the existence of the objects that they're about. For instance, if we read '3 is prime' as making a straightforward claim about the nature of the number 3, and if we allow that this sentence is literally true, then we are committed to believing in the existence of the number 3. But

[2] See, e.g., Plato's *Meno* and *Phaedo*; [Frege 1893-1903]; [Gödel 1964]; and [Quine 1948], [Quine 1951].



(5) If there are such things as mathematical objects (i.e., things that our mathematical theories are about), then they are abstract objects. For instance, if there is such a thing as the number 3, then it is an abstract object, not a physical or mental object. Therefore,
(6) There are such things as abstract mathematical objects, and our mathematical theories provide true descriptions of these things. In other words, mathematical platonism is true.

Arguments similar to this are sometimes called the ***Quine-Putnam indispensability argument***.[3] However, just about all of this argument, including the part in (1) about the applicability of mathematics, has roots in the work of Frege. In any event, when I talk in this essay about the "Quine-Putnam argument," I will have in mind only the subargument contained in (1)–(2).

There are a number of ways that anti-platonists can respond to the above argument, and different kinds of anti-platonists will respond in different ways. The most important divide in the anti-platonist camp is between the realists on the one hand and the anti-realists, or the nominalists, on the other. Nominalists reject the existence of mathematical objects like numbers and sets (or as philosophers would say, nominalists deny that mathematics has an *ontology*). So they deny that our mathematical theories provide true descriptions of some part of the world. Realistic anti-platonists, on the other hand, maintain that our mathematical theories *do* provide true descriptions of objects that exist in the world, but they deny that these objects are abstract. Thus, in connection with the above argument, realistic anti-platonists reject premise (5), and nominalists reject either (4), (3), or (2), depending on the kind of nominalism they endorse. In particular, neo-Meinongians reject (4); paraphrase nominalists reject (3); and fictionalists reject (2), as well as the argument for (2) contained in (1). Thus, the argument in (1)–(6) is set up so that as I run through the different responses to the argument, from the rejection of (5) back to the rejection of (2), we will get something of a taxonomy of the various kinds of anti-platonism. In what follows, I will run through these anti-platonist responses to the Fregean argument, indicating what (if anything) I think is wrong with the various views and responses. In particular, I will discuss realistic versions of anti-platonism in section 2.2 and nominalistic versions of anti-platonism in section 2.3.

### 2.2 Realistic Anti-Platonism

Realistic anti-platonism is the view that (a) our mathematical theories provide true descriptions of objects that exist in the world, but (b) these objects are not abstract. Now, if mathematical objects are not abstract—if, that is, they are concrete—then it seems that they must be either mental objects of some kind or (non-mental) physical objects of some kind. (One might think that they could also be social objects, or perhaps social constructions. It seems, though, that social objects would ultimately have to reduce to either mental objects or abstract objects.[4]) Thus, there

[3] See, e.g., [Quine 1948], [Quine 1951], [Putnam 1971], and [Colyvan 2001].

[4] This does not mean that we should think of social construction views of mathematics as either psychologistic or platonistic. It's easy to imagine versions of this view which are such that (a) the social objects in question could only be abstract objects, but (b) the view clearly rejects the existence of abstract objects, and so (c) in the end, the view is best thought of as a sort of anti-realism, perhaps a kind of fictionalism. I think this is probably the best way to understand Hersh's [1997] view, although I doubt that he would want to put it this way.

are two kinds of realistic anti-platonism: physicalism and psychologism. I will now discuss these in turn.

### 2.2.1 *Physicalism*

Advocates of physicalism maintain that our mathematical theories are about (non-mental) things that exist in the physical world. Thus, they agree with platonists that our mathematical theories provide true descriptions of things that exist independently of us and our thinking, but they reject the platonist idea that mathematical objects are abstract. There are a few different ways that one might try to develop a physicalist view of mathematics. The most famous strategy here is due to John Stuart Mill ([1843], book II, chapters 5 and 6). On his view, mathematics is about ordinary physical objects and it is just a very general empirical science. For instance, Mill takes arithmetical sentences like '$2 + 3 = 5$' to make very general claims about piles of objects. Thus, on this view, '$2 + 3 = 5$' does not tell us something about specific entities (numbers). Rather, it tells us that whenever we push a pile of two objects together with a pile of three objects, we get a pile of five objects—or something along these lines.

One problem with this view is that in order to account for contemporary mathematics in this general way, a contemporary Millian would have to take set theory to be about physical piles as well. This, however, is untenable. One argument here is that sets could not be piles of physical stuff, because corresponding to every physical pile—or, indeed, every individual physical object—there are infinitely many sets. Corresponding to a ball, for instance, is the set containing the ball, the set containing its molecules, the set containing that set, and so on. Clearly, these sets are not purely physical objects, because (a) they are all distinct from one another, and (b) they all share the same physical base (i.e., they're all made of the same matter and have the same spatiotemporal location). Thus, there must be something non-physical about these sets, over and above the physical base that they all share. So sets cannot be purely physical objects.

A second problem with physicalism is that it seems to imply that mathematics is an empirical science, contingent on physical facts and susceptible to empirical falsification. This seems to fly in the face of the facts about actual mathematical methodology. Of course, there's a *sense* in which mathematical thinking is sometimes "empirical": mathematicians often proceed by thinking of examples and counterexamples. But Millian physicalism implies that *all* mathematical claims—e.g., '$2 + 3 = 5$'—could in principle be falsified by discoveries about *physical* objects. This seems not just implausible (because most mathematical assertions cannot be empirically falsified by discoveries about the nature of the physical world), but also out of step with mathematical practice. It is just not plausible to interpret ordinary utterances of sentences like '$2 + 3 = 5$' as being about the physical world in a way that makes them contingent upon physical facts.

A third problem is that physicalism seems incapable of accounting for the truth of mathematical sentences that require the existence of infinitely many objects. Consider, e.g., the sentence, 'There are infinitely many transfinite cardinals.' It is hard to believe that this is a true claim about purely physical objects. (For a more thorough argument against the Millian view of mathematics, see my [1998a], chapter 5, section 5.)

Philip Kitcher [1984] has developed a view that might *seem* like a contemporary version of Millian physicalism. On Kitcher's view, our mathematical theories should be interpreted as (or



paraphrased into) sentences about the activities of an ideal agent (a creature who pushes blocks around, adding them to piles of blocks, taking them away from piles, and so on).[5] But according to Kitcher, there aren't really any such things as ideal agents, and so, on his view, our mathematical theories are vacuous—that is, in the end, they are not about anything. So Kitcher isn't really a physicalist at all; he is, rather, an anti-realist. In particular, he is what I will call a paraphrase nominalist. Thus, Kitcher's view falls prey to the argument I give below (section 2.3.2) against paraphrase nominalism.[6]

Another way to develop physicalism would be to claim that mathematical objects are properties of some kind and to adopt a physicalistic view of properties. For example, one might take natural numbers to be properties of piles of physical objects. (This sort of view has been defended by Armstrong [1978].) There are numerous problems with views like this. To name just one, they seem to encounter problems with branches of mathematics that deal with things not found in the physical world. For instance, talk of certain large transfinite cardinals is not plausibly interpreted as being about properties that are found in the physical world. One might try to interpret such talk as being about convoluted properties that *are* found in the physical world. But any such interpretation would involve a significant departure from actual mathematical practice, and it would be susceptible to the argument given below (section 2.3.2) against paraphrase nominalism.

### 2.2.2 *Psychologism*

This is the view that there do exist mathematical objects like numbers and sets but that they do not exist independently of us. Instead, they are mental objects; in particular, the claim is usually that they are something like ideas in our heads. Thus, for instance, on this view, '3 is prime' is about a certain mental object, namely, the idea of 3. Today, most philosophers of mathematics think that psychologism is completely untenable, but the view was popular in the late nineteenth and early twentieth centuries (see, e.g., the early Husserl [1891]).

(It is often thought that intuitionism is a form of psychologism, but this is a mistake. What's true is that many intuitionists—most notably, Brouwer ([1912] and [1948]), and Heyting [1956]—have also endorsed psychologism. But intuitionism is perfectly consistent with platonism and nominalism, and psychologism is consistent with a rejection of intuitionism. See my [forthcoming] for more on the difference between psychologism and intuitionism.)

We can obtain a better understanding of psychologism, and of why philosophers find the view implausible, by distinguishing two different theses that are inherent in this view. As we saw above (section 1), we can think of a philosophy of mathematics as a *semantic* theory (an empirical theory about what certain words mean) and (usually) an *ontological* theory (a theory about what sorts of things really exist). In particular, psychologism is the conjunction of a certain ontological thesis

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[5] The reason we can't take these sentences to be about actual agents is that most of the operations in question have never been performed. For instance, it seems likely that no one has ever pushed a pile of 17,312 blocks together with a pile of 8,643,912 blocks.

[6] One might argue that Kitcher's view is actually worse off than other paraphrase nominalist views, because it fails to deliver what is ordinarily taken to be the benefit of paraphrasing, namely, salvaging the truth of mathematics while claiming that our mathematical theories aren't about anything. For while some mathematical claims come out vacuously true on Kitcher's view, some of them come out untrue. Thus, his view is actually an odd sort of cross between paraphrase nominalism and fictionalism.

(namely, that there do exist such things as number-ideas, and set-ideas, and so on, in human heads) and a certain semantic thesis (namely, that sentences like '3 is prime' and '8,937,461.1174 > 67,491.398,' in ordinary mathematical discourse, are best interpreted as being *about* these ideas in our heads). Thus, taking the two claims together, psychologism is the claim that mathematical sentences like the above are *true* claims about *actual objects* that *literally exist* in human heads. Now, psychologism's ontological thesis—that we have ideas of mathematical objects in our heads—is obviously true: we do have ideas of numbers and so on. So *whatever* philosophy of mathematics you endorse, you should endorse the psychologist's ontological thesis. Thus, it is the *semantic* thesis of psychologism that really sets that view apart from other views. And it is this semantic thesis that other philosophers (i.e., platonists and physicalists and anti-realists) deny. That is, they deny that terms like '67,491.398' should be interpreted as denoting ideas in our heads.

There are a number of well-known arguments against the psychologist's semantic thesis. Many of these were articulated by Frege ([1884, introduction and section 27], [1893–1903, introduction], [1894], and [1919]). I will present three of these arguments here (although I should note that while the arguments I provide are all Fregean in spirit, my formulations are a bit different from Frege's). The first argument is that psychologism makes mathematical truths contingent upon psychological truths. So, for instance, if we all died, '4 is greater than 2' would suddenly become untrue. But this seems wrong: it seems that mathematics is true independently of us. Whether 4 is greater than 2 has nothing at all to do with how many humans are alive. Second, psychologism seems incompatible with the fact that there are infinitely many numbers, since clearly, there are not infinitely many number-ideas in human brains. (This is not to say that humans can't *conceive* of an infinite set. The point is, rather, that it's not the case that there *is* an infinite set of actual objects (i.e., distinct number-ideas) *actually residing* in human heads. Standard arithmetical theories like Peano Arithmetic (PA) imply that there is such a thing as the number 1, and there is such a thing as the number 2, and 2 is not identical to 1, and so on. Thus, psychologism together with PA implies that there actually exist infinitely many distinct number-ideas in human heads. Thus, either psychologism or PA is false. Third, psychologism suggests that the proper methodology for mathematics is that of empirical psychology. That is, if psychologism were true, the proper way to discover whether, say, there is a prime number between 10,000,000 and 10,000,020, would be to do an empirical study of humans and ascertain whether there is, in fact, an *idea* of such a number in one of our heads. But, of course, this is not the proper methodology for mathematics; the proper methodology involves mathematical proof, not empirical psychology. As Frege says [Frege 1884, section 27], "Weird and wonderful . . . are the results of taking seriously the suggestion that number is an idea."

Given the implausibility of the semantic thesis inherent in psychologism, one might ask whether there's any way to endorse a psychologistic view of mathematics without endorsing the strong semantic thesis that leads to such obvious problems. The answer is no. One might *try* to do this by taking psychologism's ontological thesis (i.e., the thesis that humans do have ideas of numbers and sets and so on) and strengthening it in an effort to generate a thesis that isn't so trivial and that would separate psychologism from platonism and anti-realism. But I think it can be argued that this is hopeless. To appreciate this point, consider the following thesis:

> **(COR)** Mathematical correctness is ultimately settled by the ideas that we have in our heads. More specifically, a mathematical sentence is correct just in case it is "built into," or follows from, the notions, conceptions, intuitions, and so on that we have in connection with the given branch of mathematics (or if we have no substantive pre-theoretic conception of the objects being studied, then correctness is determined by the axioms that we happen to be working with). Thus, for instance, an arithmetical sentence is correct if and only if it is built into our full conception of the natural numbers (FCNN), where FCNN is just the sum total of all of our "natural-number thoughts" and everything that follows from these thoughts. And a set-theoretic sentence is correct if and only if it is built into our full conception of the universe of sets, or our notion of set. And so on.[7]



This might seem, at first, like a pretty psychologistic thesis. But I have argued elsewhere [Balaguer 2001] that platonists and anti-realists can—indeed *should*—endorse (COR). I will say a few words about this below: we'll see in section 3.3.4 that the best versions of platonism imply (COR), and we'll see in section 2.3.3 that the best versions of anti-realism imply (COR). So (COR) is not a particularly psychologistic thesis. Likewise, it is not a particularly "social constructivist" thesis. It is perfectly consistent with platonism and anti-realism to claim that social and psychological facts are crucially important to the determination of mathematical truth. So, again, what separates psychologism from other more plausible views (e.g., platonism and anti-realism) is not anything like (COR); what sets psychologism apart is the *semantic* thesis that our mathematical sentences are about ideas in our heads. But as we've seen, there are strong arguments against this thesis. And this is why philosophers of mathematics roundly reject psychologism.

## 2.3 Anti-Realistic Anti-Platonism (aka Nominalism)

If arguments like the ones discussed above are correct, then sentences like '3 is prime' are *not* about physical or mental objects. So it would seem that premise (5) in the Fregean argument for platonism is true. We turn now to the nominalistic responses to the Fregean argument. For platonists, this is the hard part. There is a good deal of agreement among philosophers of mathematics that psychologism and physicalism are untenable. Hence, most philosophers of mathematics are either platonists or nominalists. But there is very little agreement as to whether platonism or nominalism is correct.

Mathematical nominalism is the view that there are no such things as mathematical objects like numbers and sets. So on this view, our mathematical theories do not provide true descriptions of some part of the world. Thus, while nominalists would admit that there are such things as piles of three stones, and thoughts of the number 3 in people's heads, they would deny that the '3' of ordinary mathematical discourse can plausibly be interpreted as denoting any of these things. Nominalists think that when we get the *correct* interpretation of the term '3,' as it's used in ordinary mathematical discourse, it turns out that there is simply no such thing as the number 3. There are at least three different major subcategories within the nominalist camp. The first, which I will call *neo-Meinongianism*, involves a rejection of premise (4) in the Fregean argument. The second, which I will call *paraphrase nominalism*, involves a rejection of premise (3). And the third, *fictionalism*, involves a rejection of premise (2) and hence also a rejection of the Quine-Putnam argument for (2), i.e., the argument contained in (1). I will now discuss these views in turn.

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[7] I assume here, and in what follows, that our various full conceptions are consistent. Anyone who wanted to endorse a view like (COR) would obviously have to tell a different story in connection with cases involving inconsistent full conceptions. I won't go into this here, but see my [2001] for more on this.

### 2.3.1 Neo-Meinongianism

Let traditional Meinongianism be the view that

(a) every *singular term*—e.g., 'Clinton,' '3,' and 'Sherlock Holmes'—denotes an object that has some sort of being (that *subsists*, or that *is*, in some sense); but
(b) only some of these objects have full-blown existence; and
(c) mathematical sentences like '3 is prime' express truths about objects that don't exist (but that still have some sort of being).

This view has been almost universally rejected. The standard argument against it (see, e.g., [Quine 1948], p. 3) is that it does not provide a view that is clearly distinct from platonism and merely creates the illusion of a different view by altering the meaning of the term 'exist.' On the standard meaning of 'exist,' any object that has any being at all exists. So according to standard usage, traditional Meinongianism implies that numbers exist. But this view clearly doesn't take such things to exist in spacetime. Therefore, traditional Meinongianism implies that numbers are abstract objects—which, of course, is just what platonism says.

There is a second version of Meinongianism, however, that is not refuted by the above argument. This view, which we can call *neo-Meinongianism*, holds that

(a) '3 is prime' should be given a face-value interpretation according to which it makes a straightforward claim about the nature of the number 3 (in particular, it says that this object is prime); and
(b) there is no such thing as the number 3 (i.e., it has no sort of being whatsoever); and yet
(c) '3 is prime' is literally true.

Thus, the idea here is that we can make true claims about things that don't exist at all. We can say (truly) of the number 3 that it is prime, even though there is no such thing as 3. (It is standardly thought that Meinong [1904] held the traditional Meinongian view described above, but one might argue that he actually endorsed what I am calling neo-Meinongianism. In any event, neo-Meinongianism has been endorsed by Richard Routley [1980]—or Richard Sylvan, as he later became known—and by Priest [2003], Azzouni ([1994], [2004]), and Bueno [2005].

One problem with neo-Meinongianism is that just as traditional Meinongians seem to alter the meaning of 'exist,' so neo-Meinongians seem to alter the meaning of 'true.' It seems that, on the standard meaning of 'true,' if there is no such thing as the object $a$, then sentences of the form '$a$ is $F$' cannot be *literally* true. In other words, if you believe that '$a$ is $F$' is literally true, then you also have to believe in the existence of the object $a$. Given this, it seems that neo-Meinongianism is untenable. If we want to maintain that mathematical sentences like '3 is prime' are literally true (and that they are of the form '$a$ is $F$'), then we have to admit that there are mathematical objects like the number 3. (See also David Lewis [1990] for an argument against Routley's view.)

Another view that might be mentioned here—a view that's related to neo-Meinongianism but also importantly different—is *conventionalism* (see, e.g., [Ayer 1946, chapter 4], [Hempel 1945], [Carnap 1934], and [Carnap 1956]). This view holds that sentences like '3 is prime' and 'There are infinitely many prime numbers' are *analytic*. That is, they are true in virtue of meaning, or linguistic conventions—along the lines of, say, 'All bachelors are unmarried' or 'All warlocks are warlocks.' But again, if sentences like '3 is prime' imply that mathematical objects like 3



exist, then it's hard to see how these sentences could be true by convention, or true in virtue of meaning. It's hard to see how the *existence* of infinitely many numbers could follow from our accepting a set of linguistic conventions—unless we're talking about some sort of psychologism, a view that we've already dispensed with.

### 2.3.2 Paraphrase Nominalism

Paraphrase nominalists reject premise (3) of the Fregean argument. That is, they claim that we should *not* read sentences like '3 is prime' at face value, i.e., as being of the form '$a$ is $F$.' Instead, they claim, sentences like this should be read as having different logical forms. In particular, paraphrase nominalists think that these sentences have logical forms that do not imply the existence of any objects. Thus, to back this claim up, paraphrase nominalists have to specify what they think sentences like '3 is prime' *are* saying. Or in other words, they have to give *paraphrases* of these sentences that reveal their real logical forms. There are several different strategies in the literature for paraphrasing mathematics. One view here, known as *if-thenism*, holds that '3 is prime' can be paraphrased by 'If there were numbers, then 3 would be prime.'[8] For an early view of this general kind, see the early Hilbert ([1899] and his letters to Frege in [Frege 1980]); for later versions, see [Putnam 1967a], [Putnam 1967b] and [Hellman 1989]. For other paraphrase views, see, e.g., [Curry 1951] and [Chihara 1990].

The main problem with paraphrase nominalist views is that they're committed to implausible empirical hypotheses about the intentions of mathematicians and ordinary folk. For instance, if-thenism is committed to the thesis that when mathematicians and ordinary folk utter sentences like '3 is prime,' what they really mean to say is that if there were numbers, then 3 would be prime. But there is no evidence for the thesis that this is what is meant, and what's more, it seems obviously false. The same point can be made in connection with all of the other versions of paraphrase nominalism. In short, it just seems wrongheaded to claim that when mathematicians and ordinary folk utter sentences like '3 is prime,' they are speaking non-literally and really *mean* to be saying something other than what they *seem* to be saying, e.g., that 3 is prime. Now, one might question whether paraphrase nominalists are really committed to the empirical hypothesis that their nominalist paraphrases capture what mathematicians really mean to be saying when they utter sentences like '3 is prime.' But it's easy to see that they *are* so committed. For if they admit that ordinary mathematical utterances should be interpreted at face-value, then their view will collapse into fictionalism (see 2.3.3). They will be committed to saying that when we interpret our mathematical theories as saying what they actually mean in the mouths of mathematicians, they imply the existence of abstract objects. But since paraphrase nominalists deny that there are any such things as abstract objects, they will have to say that our mathematical theories, interpreted literally, are not true. And this is just what fictionalism says.

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[8] One might point out here that '3 is prime' can be paraphrased by the sentence 'Aside from 3 and 1, there does not exist a pair of natural numbers, n and m, such that $n \times m = 3$.' But this sort of mathematical paraphrase is not helpful to nominalists, because many such paraphrases will still imply the existence of mathematical objects. For instance, if we switch the example to '4 is composite,' then the relevant mathematical paraphrase would be something like this: 'In addition to 4 and 1, there exists a pair of natural numbers, n and m, such that $n \times m = 4$.' But this sentence obviously implies the existence of mathematical objects. So this kind of paraphrase won't work. What nominalists need is a general method of paraphrasing that always delivers sentences that don't imply the existence of any mathematical objects.

### 2.3.3 Mathematical Fictionalism

Mathematical fictionalism (or simply fictionalism, as I'll call it) is the view that

(a) our mathematical sentences and theories should be interpreted at face value; (e.g., '3 is prime' should be interpreted as making—or purporting to make—a straightforward claim about the number 3); but
(b) there are no such things as abstract objects such as the number 3; and so
(c) mathematical sentences like '3 is prime' are not true.

Or, equivalently, we can say that fictionalists endorse the *semantic* theory of platonists but reject the *ontological* theory of platonists. Thus, according to fictionalists, our mathematical theories are not literally true for the same reason that, say, *Alice in Wonderland* is not literally true. Just as there are no such things as talking rabbits and hookah-smoking caterpillars and so on, so too there are no such things as numbers and sets and so on. Fictionalists agree with platonists that premises (3)–(5) in the Fregean argument are correct. They admit that if our mathematical theories are true, then there are abstract objects and platonism is correct. But fictionalists reject (2), and they reject the Quine-Putnam argument for (2)—i.e., the argument contained in (1).[9] Fictionalism was first introduced by Hartry Field ([1980], [1989]) (see below). He saw the view as being wedded to the thesis that empirical science can be nominalized. That is, on Field's view, scientific theories can be restated so that they don't contain any reference to, or quantification over, mathematical objects. In my [1996a] and [1998a], I defend a version of fictionalism that is divorced from the nominalization program, and similar versions of fictionalism have been endorsed by Rosen [2001] and Yablo [2002].

The most important objection to fictionalism is the Quine-Putnam indispensability objection. The argument here can be put like this:

> Our mathematical theories are extremely useful in empirical science; indeed, they seem to be indispensable to our empirical theories. Therefore, assuming that we want to claim that our empirical theories provide accurate pictures of the world, it seems that we also have to maintain that our mathematical theories provide accurate descriptions of the world. Therefore, it seems that fictionalism is false.

Before saying how fictionalists can respond to this argument, it should be noted that fictionalism is not the only philosophy of mathematics that encounters a problem here. Every philosophy of mathematics has to account for the fact that mathematics is applicable (and perhaps indispensable) to empirical science. And for most of the standard views in the literature, there is some initial reason to think that they might not be able to provide the required explanation. The only exception here is physicalism (a view that, as we've seen, is untenable for other reasons). Indeed, I think it can be argued that the problem of applications is essentially equivalent for all non-physicalistic

[9] In addition to fictionalism, there is a second (much more radical) way to claim that our mathematical theories are not true and, hence, that (2) is false. One could endorse a *non-cognitivist* view of mathematics, claiming that sentences like '3 is prime' don't really say anything at all and, hence, aren't the sorts of things that have truth values. One such view is *game formalism*, which holds that mathematics is a game of symbol manipulation. According to this view, '3 is prime' is one of the "legal results" of the game of arithmetic. This view was defended by Heine and Thomae and attacked vigorously by Frege (see Frege [1893–1903], sections 88–131). One might also interpret Wittgenstein's [1956] philosophy of mathematics as non-cognitivist, although this is controversial.



views of mathematics. And I also think it can be argued that either all of these views can solve the problem or else none of them can. In what follows, I will discuss the question of how fictionalists can respond to this worry; but the view of applications and indispensability that I favor can be conjoined with other views as well—most notably, with platonism and other (non-fictionalistic) versions of nominalism.

Fictionalists have developed two different responses to the Quine-Putnam argument. The first was developed by Field [1980]. He argues that

(a) mathematics is in fact not indispensable to empirical science; and
(b) the fact that it is applicable to empirical science in a dispensable way can be explained without abandoning fictionalism.

Claim (b) is fairly plausible and has not been subjected to much criticism,[10] but claim (a) is highly controversial. In order to establish thesis (a), we would have to argue that all of our empirical theories can be *nominalized*, i.e., reformulated in a way that avoids reference to, and existential quantification over, abstract objects. Field [1980] tried to motivate this by carrying out the nominalization for one empirical theory, namely, Newtonian Gravitation Theory. And in my [1996b] and [1998a], I show how to extend Field's strategy to quantum mechanics. However, philosophers have raised several objections to Field's nominalization program—see, e.g., [Malament 1982], [Resnik 1985], and [Chihara 1990, chapter 8, section 5]. The consensus opinion seems to be that this program cannot be made to work, although this is far from established.

The second fictionalist response to the Quine-Putnam argument is

(a) to grant for the sake of argument that mathematics is hopelessly and inextricably woven into some of our empirical theories; and
(b) to simply *account* for these indispensable applications from a fictionalist point of view.

I developed this strategy in my [1996a], [1998b], and [1998a, chapter 7]; the idea has also been pursued by Rosen [2001] and Yablo [2002]. The central ideas are as follows. Because abstract objects are entirely non-causal, and because our empirical theories don't assign any causal role to abstract objects, it follows that the truth of empirical science depends upon two sets of facts that are entirely independent of one another. That is, the two sets of facts hold or don't hold independently of one another. One of these sets of facts is purely platonistic and mathematical, and the other is purely physical (or more precisely, purely nominalistic). Consider, for instance, the sentence

(A) The physical system S is forty degrees Celsius.

This sentence says that the physical system S stands in the Celsius relation to the number 40. But, trivially, it does not assign any causal role to the number 40. It is not saying that the number 40 is *responsible* in some way for the fact that S has the temperature it has. Rather, what's going on here is that we are using the numeral '40' to help us say what we want to say about S. In essence, what we're doing is using '40' as a *name* of a certain temperature state. (It is *convenient* to use numerals here, instead of ordinary names like 'Ralph' and 'Jane,' because the real numbers are structured in the same way that the possible temperature states are structured.) Thus, given

[10] See, however, [Shapiro 1983a] for one objection; and for a response, see [Field 1989, essay 4].

this, it follows that if (A) is true, it is true in virtue of facts about S and 40 that are entirely independent of one another. And the same point seems to hold for all of empirical science. Since no abstract objects are causally relevant to the physical world, and since empirical science never says that they are, it follows that if empirical science is true, then its truth depends upon two entirely independent sets of facts: a set of purely physical or nominalistic facts and a set of purely platonistic facts.

Now, since these two sets of facts hold or don't hold independently of one another, it could be that (a) there does exist a set of purely physical facts of the sort required for the truth of empirical science, but (b) there *doesn't* exist a set of purely platonistic facts of the sort required for the truth of empirical science (because there are no such things as abstract objects). Therefore, mathematical fictionalism is perfectly consistent with the claim that empirical science paints an essentially accurate picture of the physical world. In other words, fictionalists can endorse what I call ***nominalistic scientific realism***. This is just the view that (a) and (b) above are true. In other words, it's the view that the physical world holds up *its end* of the "empirical-science bargain." (This view is different from standard scientific realism, because it doesn't imply that our empirical theories are *strictly* true. Nonetheless, this view is still a realist view, for according to this view, the physical world is *essentially just the way empirical science makes it out to be*. After all, this view says that there does exist a set of purely physical facts of the sort needed for the truth of empirical science.) Therefore, fictionalism is consistent with whatever role mathematics plays in empirical science, indispensable or not. For even if mathematics can't be eliminated from our empirical theories, and even if there are no such things as mathematical objects (and hence our empirical theories aren't literally true), the picture that empirical science paints of the physical world could still be essentially accurate.

Now, one might wonder what mathematics is doing in empirical science, if it doesn't need to be true in order for empirical science to be essentially accurate. The answer, I think, is that mathematics appears in empirical science as a *descriptive aid*. That is, it provides us with an easy way of saying what we want to say about the physical world.[11] For a more complete formulation of this second fictionalist response to the Quine-Putnam argument, see my [1996a], [1998b], and [1998a, chapter 7].

Even if fictionalists can successfully respond to the Quine-Putnam argument in this way, they are not out of the woods. For while the Quine-Putnam argument is the most important worry about fictionalism, there are other objections that one might raise against the view. I do not have the space to discuss all of these objections here, or to respond in full to the ones I do discuss, but I would like to say a few words about some of the more obvious of these objections.

**Objection 1:** *Fictionalism seems incapable of accounting for the objectivity of mathematics. In particular, it seems inconsistent with the fact that there is an important difference between sentences like '3 is prime' on the one hand and sentences like '3 is composite' on the other. It seems that the difference here is that '3 is prime' is true whereas '3 is composite' is false. But fictionalists can't say this, because they think that both of these sentences are untrue. So what* can *fictionalists say about this?*

[11] One might wonder how it could be that mathematics is indispensable to an empirical theory T if the mathematics in T functions merely as a descriptive aid in that theory. The answer is that it may be impossible to formulate a theory that doesn't use any mathematics and yet still counts as a "version of T." There might be theories that don't use any mathematics and that are *empirically equivalent* to T in the sense that they imply the same predictions about the physical world. But they might be so unlike T, in "look and feel," that it would be implausible to treat them as "alternate versions of T."



*Response*: As Field [1980] pointed out when he first introduced the view, fictionalists can say that the difference between '3 is prime' and '3 is composite' is analogous to the difference between 'Oliver Twist lived in London' and 'Oliver Twist lived in sin.' In other words, the difference is that '3 is prime' is part of a certain well-known mathematical story, whereas '3 is composite' is not. We might express this idea by saying that while neither '3 is prime' nor '3 is composite' is literally true, there is another truth predicate (or pseudo-truth predicate, as the case may be)—viz., 'is true in the story of mathematics'—that applies to '3 is prime' but not to '3 is composite.' This seems to be the view that Field endorses, but there is more that needs to be said on the topic. According to fictionalism, there are alternative mathematical "stories" consisting of sentences that are not part of standard mathematics. Thus, the real difference between '3 is prime' and '3 is composite' is that the former is part of *our* story of mathematics, whereas the latter is not. (Of course, there is no *consistent* mathematical story that contains the sentence '3 is composite' and in which that sentence means what it does in English; but that's not relevant here.)

**Objection 2:** *OK, this will enable fictionalists to account for the difference between '3 is prime' and '3 is composite,' but there is more than this to the objectivity of mathematics. For instance, it could turn out that mathematicians are going to* discover *an objectively correct answer to the question of whether the continuum hypothesis (CH) is true or false; but it's not clear how fictionalists could account for this. Given that both CH and its negation (~CH) are consistent with the standard Zermelo-Fraenkel axiomatization of set theory (ZF), how could fictionalists claim that one of these sentences is true in the story of mathematics whereas the other is not?*

*Response*: In order to respond to this objection, fictionalists need to get more precise about what determines whether a sentence is true in the story of mathematics, or part of our story of mathematics. According to the version of fictionalism I favor, our story of mathematics goes beyond the axioms systems that we currently accept. It covers what I call the *full conceptions* that we have of the objects, or purported objects, in the various branches of mathematics. That is, it covers the sum total of the intentions that we, as a community, have regarding those objects. So the story of arithmetic includes everything that follows from our full conception of the natural numbers. And the story of set theory includes everything that follows from our full conception of the universe of sets, or our notion of set. And if we have no substantive pretheoretic conception of the objects being studied, then the given "full conception" is exhausted by the axiom system in question.

Given this, fictionalists can account for how mathematicians could *discover* an objectively correct answer to the CH question. Suppose, for instance, that some mathematician thought of some new axiom candidate A such that (i) all mathematicians agreed that A was intuitively obvious, and (ii) ZF + A implied CH. Then mathematicians would claim that we had discovered that CH was correct (and that it had been correct all along, that we hadn't just made this up). Fictionalists of the kind I have in mind can account for this. They can claim that the fact that A was intuitively obvious to all mathematicians shows that it was inherent in, or followed from, our notion of set, and hence that CH followed from our notion of set (even before we discovered A). So fictionalists can claim in this case that CH was part of the story of mathematics all along, even though we hadn't noticed this.

Now, this is not to say that fictionalists are committed to saying that there *is* an objectively correct answer to the CH question. In fact, they're not—fictionalism of this kind is consistent

with the claim that it might be that there is no objectively correct answer to the CH question. For it *may* be that neither CH nor ~CH follows from our notion of set. In this case, according to my version of fictionalism, neither CH nor ~CH would be true in the story of mathematics. And so there would be no objectively correct answer to the CH question. Fictionalism is thus consistent with whatever mathematicians end up saying about CH. And this, I think, is a very attractive feature of fictionalism. For the question of what we ought to say about CH is a *mathematical* question. We don't want our philosophy of mathematics dictating what mathematicians ought to say about this.[12]

(By the way, the above considerations suggest that fictionalists should endorse (COR) (see section 2.2.2). From this it follows that fictionalism has more in common with "social constructivist" views than one might have thought. Below, we will see that the best versions of platonism also imply (COR), and so they too have something in common with social constructivist views. Indeed, depending on what is meant by "social constructivism," one might even conclude that, surprisingly, the best versions of platonism and fictionalism *are* social constructivist views. But if "social constructivism" is taken to involve a non-standard view of the *meanings* of mathematical sentences, then these views are not social constructivist views. For according to both platonism and fictionalism, our mathematical sentences and theories should be read at face value, as being about abstract mathematical objects (or as fictionalists would put it, as *purporting* to be about such objects).)

I have just scratched the surface of the topic of fictionalism and objectivity. There is much more to say about this. I cannot say any more about it here, but I have discussed it at length elsewhere, in my [2001] and my [1998a].

**Objection 3:** *Fictionalism is just wildly implausible on its face. Mathematics and fiction are radically different enterprises; there are numerous obvious disanalogies between the two.*

*Response*: Fictionalists can simply grant that there are deep and important disanalogies between mathematics and fiction, because they aren't committed to there being any deep similarities between the two enterprises. Mathematical fictionalism is a view about mathematics only. It doesn't say anything at all about fictional discourse, and so it is not committed to there being any deep similarities between mathematics and fiction. Likewise, fictionalists aren't committed to there being any deep similarities between mathematics and metaphor, or between mathematics and anything else. (For this reason, the name 'fictionalism' might be a bit misleading; a less misleading name might be 'lack-of-reference-ism,' or 'not-true-ism.')

Let's summarize what we've found so far. There are two realistic alternatives to platonism, namely, physicalism and psychologism. And there are three main anti-realist alternatives to platonism, namely, neo-Meinongianism, paraphrase nominalism, and fictionalism. I have argued (albeit rather briefly) that, aside from fictionalism, none of these views is tenable. Thus, returning to the Fregean argument for platonism with which we began, it seems that premises (3)–(5) are correct. This means that if (2) is true then (6) is also true. That is, it means that if our mathematical

[12] The remarks in the text suggest that fictionalists should say that a mathematical sentence is "correct" iff it follows from the full conception that we have of the objects, or purported objects, in the given branch of mathematics. But this isn't quite right, for if the given full conception is *inconsistent*, fictionalists need to say something else. See my [2001, note 5] for more on this.



theories are literally true then platonism is true. But my own view is that we don't have any good reason to believe that our mathematical theories are literally true; in particular, I don't think the Quine-Putnam argument contained in (1) gives us a good reason to believe (2). As we will presently see, however, this is *not* to say that I think we have good reason to endorse fictionalism.

## 3 The Epistemological Argument Against Platonism

There are a number of arguments against platonism in the literature, but one of these arguments stands out as the strongest, namely, the epistemological argument. This argument goes all the way back to Plato, but it has received renewed interest since 1973, when Paul Benacerraf presented a version of the argument. The argument can be put in the following way (see my [1998a]):

(E1) Human beings exist entirely within spacetime.
(E2) If there exist any abstract mathematical objects, then they exist outside of spacetime. Therefore, it seems very plausible that
(E3) If there exist any abstract mathematical objects, then human beings could not attain knowledge of them. Therefore,
(E4) If mathematical platonism is correct, then human beings could not attain mathematical knowledge.
(E5) Human beings have mathematical knowledge. Therefore,
(E6) Mathematical platonism is not correct.

The argument for (E3) is everything here. If it can be established, then so can (E6), because (E3) trivially implies (E4), (E5) is beyond doubt, and (E4) and (E5) trivially imply (E6). Now, (E1) and (E2) do not strictly imply (E3), and so there is room for platonists to maneuver here. And as we'll see, this is precisely how most platonists have responded. However, it is important to notice that (E1) and (E2) *seem* to provide strong motivation for (E3). They seem to imply that mathematical objects (if there are such things) are totally inaccessible to us, i.e., that information cannot pass from mathematical objects to human beings. But given this, it's hard to see how human beings could acquire knowledge of mathematical objects. Thus, we should think of this argument not as *refuting* platonism but as issuing a challenge to platonists to explain how human beings could acquire knowledge of abstract mathematical objects.

There are three strategies that platonists can use in trying to respond to this argument, and I will now discuss these in turn.

### 3.1 Rejecting the View that the Human Mind is Purely Physical

First, platonists can try to argue that (E1) is false and that the human mind is capable of somehow forging contact with abstract objects and thereby acquiring information about them. This strategy has been pursued by Plato (see *The Meno* and *The Phaedo*) and Gödel [1964]. Plato's idea is that our immaterial souls acquired knowledge of abstract objects before we were born and that mathematical learning is really just a process of coming to remember what we knew before we were born. On Gödel's version of the view, we acquire knowledge of abstract objects in much the same way that we acquire knowledge of concrete physical objects. Just as we acquire information about physical objects via the faculty of sense perception, so we acquire information about abstract objects by means of a faculty of *mathematical intuition*. Now, other philosophers have

endorsed the idea that we possess a faculty of mathematical intuition. But Gödel's version of this view—and he seems to be alone in this—seems to involve the idea that the mind is non-physical in some sense and that we are capable of forging contact with, and acquiring information from, abstract objects.[13] This view has been almost universally rejected. One problem is that denying (E1) doesn't seem to help. The idea of an immaterial mind receiving information from an abstract object seems just as mysterious and confused as the idea of a physical brain receiving information from an abstract object.

## 3.2 Rejecting the Thesis that Abstract Mathematical Objects Exist Outside of Spacetime

The second strategy that one might pursue in responding to the epistemological argument is to argue that (E2) is false and that human beings can acquire information about mathematical objects via ordinary perceptual means. The early Maddy [1990] pursued this idea in connection with set theory, claiming that sets of physical objects can be taken to exist in spacetime and, hence, that we can perceive them. For instance, on Maddy's view, if there are two books on a table, then the set containing these books exists on the table, in the same place that the books exist, and we can see the set and acquire information about it in this way. Now, according to the definitions I've been using here, views like Maddy's—i.e., views that reject (E2)—are not versions of platonism at all, because they do not take mathematical objects to be non-spatiotemporal. Nonetheless, there is some rationale for thinking of Maddy's view as a sort of non-traditional platonism. First of all, Maddy's view implies that there is an infinity of sets associated with every ordinary physical object, all sharing the same spatiotemporal location and the same physical matter. (For example, corresponding to a book, there is the set containing the book, the set containing that set, the pair containing those two sets, and so on and so forth.) But since these sets all share the same *physical base* (i.e., the same location and matter), and since they are all distinct objects, Maddy has to allow that they differ from one another in some non-physical way. Hence, on Maddy's view, there must be something about these sets that is non-physical, or abstract, in some sense of these terms. Moreover, if Maddy *didn't* take this line, her view would be untenable, because it would collapse into a version of physicalism along the lines of Mill's view, which we've already rejected. In any event, regardless of whether Maddy's view counts as version of "platonism," it is an available response to the above epistemological argument.

Maddy's early view has been subjected to much criticism, including arguments from the later Maddy [1997]. Others to attack the view include Lavine [1992], Dieterle and Shapiro [1993], myself ([1994], [1998a]), Milne [1994], Riskin [1994], and Carson [1996]. One strategy here is to argue as follows:

(a) there is more to an early-Maddian set than the aggregate of physical stuff with which it shares its location (in particular, as we've seen, there is something abstract about the set, over and above the physical aggregate); but

[13] This interpretation of Gödel is a bit controversial. Evidence for it comes not just from his [1964], but also from his [1951]. See my [1998a, section 4.2] for a discussion.



(b) human beings don't receive any sensory data about any such sets that go beyond the data that they receive about physical aggregates; therefore,
(c) there is still an unexplained epistemic gap between the information we receive in sense perception and the relevant facts about sets.

In fact, one might push this line of thought a bit further and argue as follows:

(a) traditional platonists can grant that humans receive sensory information about physical aggregates; and
(b) traditional platonists can also claim that humans can use this information in coming to knowledge of sets; therefore,
(c) Maddian platonists are no better off here, epistemologically speaking, than traditional platonists.

For a fuller version of this argument, see my [1998a].

## 3.3 Explaining How We Could Have Knowledge of Abstract Mathematical Objects Without Any Contact With Such Objects

The third and final strategy that platonists can pursue is to accept (E1) and (E2) and explain why (E3) is nonetheless false. This strategy is different from the first two in that it doesn't involve the postulation of an information-transferring contact between human beings and abstract objects. The idea here is to grant that human beings do not have any such contact with abstract objects and to explain how they can nonetheless acquire knowledge of such objects. This has been the most popular strategy among contemporary platonists. Its advocates include Quine [1951, section 6], Steiner [1975, chapter 4], Parsons ([1980], [1994]), Katz ([1981], [1998]), Resnik ([1982], [1997]), Wright [1983], Lewis [1986, section 2.4], Hale [1987], Shapiro ([1989], [1997]), myself ([1995], [1998a]), and Linsky and Zalta [1995]. There are several different versions of this view. We will look very briefly at the most prominent of them.

### 3.3.1 Justification via Empirical Confirmation?

One version of the third strategy, implicit in the writings of Quine [1951, section 6] and developed by Steiner [1975, chapter four, especially section IV] and Resnik [1997, chapter 7], is to argue as follows:

(a) our mathematical theories are embedded in our empirical theories; and
(b) these empirical theories (including their mathematical parts) have been confirmed by empirical evidence; moreover,
(c) when an empirical theory is confirmed by empirical evidence, the *entire* theory is confirmed; therefore,
(d) even though we don't have any contact with mathematical objects, we have empirical evidence for believing that our mathematical theories are true and hence for believing that there do exist abstract mathematical objects.

Given what I said above about how mathematical fictionalists can account for the role that mathematics plays in empirical science, this view seems implausible. Since abstract objects do not enter into any causal relations with anything in the physical world, it follows that we humans

would receive the same perceptual information—i.e., we would have the same set of empirical data—whether there were any such things as mathematical objects or not. Thus, empirical data can provide reason for believing only that there exist purely physical facts of the sort needed for the truth of empirical science. Empirical data do *not* provide any good reasons for believing that our empirical theories (including their implications about the existence of mathematical objects) are literally true.

A second problem with the Quine-Steiner-Resnik view is that it leaves unexplained the fact that mathematicians acquire knowledge of their theories before these theories are applied in empirical science. (For a more complete argument against the Quinean view, see my [1998a].)

### 3.3.2 *Justification via Necessity?*

A second version of the third strategy, developed by Katz ([1981], [1998]) and Lewis [1986, section 2.4], is to argue that we can know that our mathematical theories are true, without any sort of information-transferring contact with mathematical objects, because these theories are *necessarily* true. It may be that in order to know that fire engines are red, we need some sort of information-transferring contact with fire engines. But according to the Katz-Lewis view, we don't need any such contact with the number 3 in order to know that it's prime, because it couldn't have been composite, i.e., because the sentence '3 is prime' is *necessarily* true. For criticisms of this view, see Field [1989, pp. 233–38] and my [1998a, chapter 2, section 6.4]. One problem here is that there doesn't seem to be any epistemologically relevant sense in which our mathematical theories are necessarily true. Since sentences like '3 is prime' and 'There is a null set' assert that certain objects exist, they don't seem to be logically or conceptually necessary. (One might try to argue that they're *metaphysically* necessary, but there are serious problems with this suggestion. See my [1998a, chapter 2] for more on this.)

### 3.3.3 *Structuralism*

A third version of the third strategy has been developed by Resnik [1997] and Shapiro [1997]. Both of these philosophers endorse (platonistic) *structuralism*, a view that holds that our mathematical theories provide true descriptions of mathematical structures, which, according to this view, are abstract. Moreover, Resnik and Shapiro both claim that human beings can acquire knowledge of mathematical structures (without coming into any sort of information-transferring contact with such things) by simply constructing mathematical axiom systems. For these axiom systems provide *implicit definitions* of structures. There are a few different problems with this view. I discuss these problems in my [1998a], but I will just mention one of them here: Resnik-Shapiro structuralism doesn't explain how human beings could know *which* of the various axiom systems that we might formulate actually pick out real structures that exist in the mathematical realm.

### 3.3.4 *Full-Blooded Platonism*

A fourth and final version of the third strategy, developed in my own writings (see my [1992], [1995], and [1998a], and see Linsky and Zalta [1995] for a related view), is based upon the adoption of a particular version of platonism that can be called *plenitudinous platonism*, or as I call it, *full-blooded platonism* (FBP). FBP can be intuitively but sloppily expressed with the slogan, 'All possible mathematical objects exist.' More precisely, the view is that all the mathematical objects that possibly *could* exist actually *do* exist. I argue that if platonists endorse



FBP, then they can explain how human beings could acquire knowledge of abstract mathematical objects without the aid of any sort of information-transferring contact with such objects. If FBP is true, then all consistent purely mathematical theories accurately describe some collection of abstract mathematical objects. Thus, to attain knowledge of abstract mathematical objects, ***all we need to do*** is acquire knowledge that some purely mathematical theory is *consistent*. (It doesn't matter how we come up with the theory; some creative mathematician might simply "dream it up.") But knowledge of the consistency of a theory doesn't require any sort of contact with, or access to, the objects that the theory is about. Thus, the epistemological problem has been solved. We can acquire knowledge of abstract mathematical objects without the aid of any sort of information-transferring contact with such objects.

There are a number of objections that one might raise against FBP and the above FBP-based epistemology. Here, for instance, are six different objections that one might raise:

1. Your view seems to assume that humans are capable of *thinking about* abstract objects, or *referring to* them, or *formulating theories about* them. But it's not clear how humans could do these things.
2. The above sketch of your epistemology seems to assume that it will be easy for FBP-ists to account for how human beings could (without the aid of any contact with mathematical objects) acquire knowledge that certain mathematical theories are *consistent*. But it's not clear how FBP-ists could do this.
3. You may be right that if FBP is true, then all consistent purely mathematical theories truly describe *some* collection of mathematical objects, or some part of the mathematical realm. But *which* part? How do we know that it will be true of the part of the mathematical realm that its authors intended to characterize? Indeed, it seems mistaken to think that such theories will characterize *unique* parts of the mathematical realm at all. (For instance, if FBP is true, then there are infinitely many $\omega$–sequences in the mathematical realm. Can FBP-ists maintain that some unique one of these sequences is *the* sequence of natural numbers?)
4. All your theory can explain is how humans could know that *if* FBP is true, then our mathematical theories truly describe parts of the mathematical realm. It doesn't explain how humans could acquire genuine knowledge of the mathematical realm, because it doesn't explain how humans could know that FBP is true.
5. How can FBP-ists account for the applications of mathematics to empirical science? FBP implies that our mathematical theories are about objects that are causally isolated from the physical world. So why do our physical theories make use of these mathematical theories?
6. FBP seems to be inconsistent with the *objectivity* of mathematics. It seems to imply that, for example, the continuum hypothesis (CH) has no determinate truth value because CH and ~CH both accurately describe parts of the mathematical realm. Indeed, one might argue that because of this, FBP leads to the contradictory result that CH and ~CH are both true.

I respond to all of these objections, as well as a few others, in my [1998a] and my [2001]. Moreover, in my [forthcoming], I respond to some objections that have been put forward recently by other philosophers, most notably Colyvan and Zalta [1999] and Restall [2003]. I do not have the space to address these objections here, but it is worth noting, in connection with

objection 6, that the FBP-ist account of mathematical objectivity is (surprisingly) virtually identical to the fictionalist account of objectivity. According to FBP, whether CH (or any other mathematical sentence) is *true* (not just true in some model, or some part of the mathematical realm) depends on whether it's true in the *intended model* (or more precisely, in *all* intended models). But given this, it can be argued that FBP implies that a mathematical sentence is true if and only if it follows from *our intentions*, or from the full conception that *we have* of the objects in the given branch of mathematics. (And it should be noted that if we have no substantive pretheoretic conception of the objects being studied, then the given "full conception" is exhausted by the axiom system in question.) What this means is that FBP-ists are going to endorse the thesis that I called (COR) in section 2.2.2. They will not take (COR) to provide a *definition* of mathematical correctness. Like all platonists, they think that, by definition, correctness has to do with accurately describing the intended objects. But on the FBP-ist view, it *turns out* that (COR) is true.

So according to FBP, mathematical truth is ultimately determined by what follows from our "full conceptions." For instance, if CH follows from our full conception of the universe of sets, or our notion of set, then it will be true in all intended parts of the mathematical realm, and hence it will be *true*. Similarly, if ~CH follows from this full conception, then it will be true and, hence, CH will be false. And if neither CH nor ~CH follows from our full conception of the universe of sets, then they will both be true in some intended hierarchies and false in others. In this case, there will be no fact of the matter as to whether either of them is true or false. But this is essentially equivalent to what fictionalists say about how mathematical correctness—or truth in the story of mathematics—is ultimately determined. (For a full discussion of this issue, see my [2001].)

There are many other appealing features of FBP, aside from the fact that it enables platonists to solve the epistemological problem with their view. For instance, as I have argued elsewhere ([Balaguer 1998a], [Balaguer 2001]), it is only by adopting FBP that platonists can provide a plausible account of how our mathematical intuitions could be accurate indicators of mathematical truth.

In sum, then, the epistemological argument against platonism is not entirely successful. It succeeds in refuting all traditional versions of platonism, but it does not refute FBP.

## 4 Concluding Remarks

So we seem to be left with just one version of platonism (namely, FBP) and one version of anti-platonism (namely, fictionalism). Now, I have argued elsewhere ([Balaguer 1998a], [Balaguer 2001]) that FBP and fictionalism (interestingly and surprisingly) agree on almost everything about the interpretation of mathematical practice. I just said a few words about this in connection with the question of what ultimately determines mathematical truth. But it turns out that FBP and fictionalism agree on much more than this. It can be argued that FBP-ists and fictionalists should say essentially the same things in response to virtually *all* questions about mathematical practice. The reason for this is two-pronged. First, FBP-ists think that mathematical objects are causally inert, so that the existence or nonexistence of mathematical objects is irrelevant to the practice of mathematics. And second, FBP-ists think that every consistent purely mathematical theory accurately describes some collection of mathematical objects, so that, like fictionalists, they are committed to the thesis that from a purely *ontological* point of view, all consistent purely mathematical theories are equally good. Because of these two points, FBP-ists end up agreeing with fictionalists on almost all important questions about mathematical practice. They agree on questions about how and why mathematics is applicable to empirical science, what mathematical knowledge ultimately consists in, the semantics of mathematical discourse, the roles of creation and discovery in mathematics, and many other things. In short, FBP and fictionalism offer almost identical views of mathematics. The only questions about which they disagree are the question of whether there actually exist any abstract mathematical objects and (as a result) the question of whether our mathematical theories are literally true. That is, they disagree on the question of whether the mathematical statements that we all agree are *good* (or *correct*, or *acceptable*) are distinguished by being *literally true* or *true in the story of mathematics*.



Now, in my [1998a], I argue that while FBP and fictionalism can be defended against all of the standard objections to those views in the literature, there are no good positive arguments for either view. That is, there are no good arguments for the claim that there are abstract objects and hence that our mathematical theories are literally true, and there are no good arguments for the claim that there are no abstract objects and hence that our mathematical theories are strictly speaking untrue. Indeed, I argue that for a variety of reasons (most notably because we cannot obtain any information about whether there are any mathematical objects), we could *never* have any good reason to endorse FBP over fictionalism or vice versa. And finally, I argue that there is actually no fact of the matter about whether FBP or fictionalism is true, because there is no fact of the matter about whether there are any such things as abstract objects.

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