STAIRCASE TILINGS AND k-CATALAN STRUCTURES

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Abstract

Many interesting combinatorial objects are enumerated by the k-Catalan numbers, one possible generalization of the Catalan numbers. We will present a new combinatorial object that is enumerated by the k-Catalan numbers, staircase tilings. We give a bijection between staircase tilings and k-good paths, and between k-good paths and k-ary trees. In addition, we enumerate k-ary paths according to DD, UDU, and UU, and connect these statistics for k-ary paths to statistics for the staircase tilings. Using the given bijections, we enumerate statistics on the staircase tilings, and obtain connections with Catalan numbers for special values of k. The second part of the paper lists a sampling of other combinatorial structures that are enumerated by the k-Catalan numbers. Many of the proofs generalize from those for the Catalan structures that are being generalized, but we provide one proof that is not a straightforward generalization. We propose a web site repository for these structures, similar to those maintained by Richard Stanley for the Catalan numbers [33] and by Robert Sulanke for the Delannoy numbers [34]. On the website, we list additional combinatorial objects, together with hints on how to show that they are indeed enumerated by the k-Catalan numbers.

Key words: k-ary tree, k-ary number, k-Catalan number, set-valued Young tableau, staircase tiling.

AMS Mathematical Subject Classifications: 05A05, 05A15, 05C05.

1. Introduction

One of the frequently occurring sequences in combinatorics are the Catalan numbers, defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$ (A000108, [30]), which among other combinatorial structures, count binary trees ([30, 32]). Probably the most important generalization of the Catalan numbers are the *k-ary numbers*, defined by $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}$ for any positive integers k and n. Clearly, $C_n^2 = C_n$. Most of the structures known to be enumerated by the k-ary numbers are generalizations of combinatorial objects enumerated by the Catalan numbers. The most famous such structures are the *k-ary trees* [6, 12, 16, 17, 24, 31, 41], which are ordered trees in which each node has either out degree 0 or k (see A059967 and A059968 for $k = 9, 10$, [30]). Most papers on k-ary trees deal with generating, ordering and ranking k-ary trees (see for example [1, 10, 11, 14, 26, 27, 29, 38]), where ordering typically involves showing the existence of a one-to-one correspondence between k-ary trees and some other combinatorial object, for example, subdivisions of convex polygons or k-good paths ([14]). Other papers focus on enumerating statistics on k-ary trees (see for example [16, 18, 19, 22, 29, 36, 39, 41]).

A special focus have been 3-ary or ternary trees, with bijections for example to even trees, non-crossing trees, and diagonally convex ployominoes [9, 15, 23, 25]. The strong interest in k-ary trees stems from applications to computer science, for example information theory [28] and data structure design [37]. Generalizations of the Catalan numbers also show up in algebraic combinatorics, for example enumeration of quotient sets of quasi-symmetric polynomials [4] and in conjunction with Weyl groups [2, 3].

Recently, Heubach and Mansour [13] exhibited a new Catalan structure, tilings of a staircase in the $\mathbb{R}^2$ plane, and showed bijections not only to Dyck paths, but also to Motzkin paths and the Schröder paths. In Section 2, we define a generalization, tilings of the staircase $\mathcal{A}_n^k$ in the $\mathbb{R}^2$ plane, and provide a bijection to k-good paths, as well as a direct bijection between k-ary paths and k-ary trees. We conclude the section by enumerating statistics on k-ary paths and the associated statistics of the staircase tilings, and exhibit connections to the Catalan sequence.

In Section 3, we will provide a sampling of currently known and new combinatorial structures enumerated by the k-ary numbers. A larger list of such objects, some of which are known in the folklore and some of which are new, is posted on our website, www.math.haifa.ac.il/toufik/enum2005.html (click on k-ary structures). For each of the structures, we either provide a reference or an outline of the proof that shows that the structure is enumerated by C_n^k. Some of the proofs are straightforward generalizations of the respective Catalan structures. However, we will give one proof which is not a straightforward generalization.

2. Staircase tilings and k-ary trees

In this section, we define the new combinatorial structure, a tiling of the staircase $\mathcal{A}_n$ in the $\mathbb{R}^2$ plane, and then create bijections from these tilings to k-ary paths. In addition, we give a direct bijection between k-ary trees and k-ary paths. The latter bijection then allows us to study statistics on staircase tilings.

Before introducing the staircase structure, we will give some basic definitions. A *lattice path of length* n is a sequence of points $P_1, P_2, \cdots, P_n$ with $n \geq 1$ such that each point P_i belongs to the plane integer lattice and consecutive points P_i and P_{i+1} are connected by a line segment. We will consider lattice paths in $\mathbb{Z}^2$ whose permitted step types are east-steps $E = (1, 0)$ and north-steps $N = (0, 1)$. We will focus on paths that start at the origin and return to the line $y = (k-1)x$ with $k \geq 2$, and lie weakly below this line. Such a path is called a *k-good path*. The *length* of a k-good path of n east-steps and $(k-1)n$ north-steps from $(0, 0)$ to $(n, (k-1)n)$ is given by its number of steps, kn. The set of k-good paths of length kn is denoted by G_n^k (see Section 3, Structure 2, for a list of the 3-good paths of length 9). In [14], Hilton and Pedersen stated that the number of k-good paths of length kn is enumerated by

$$C_n^k = \frac{1}{n}\binom{kn}{n-1} = \frac{1}{kn+1}\binom{kn+1}{n}.$$

Note that this result can also be derived from a result given by Mohanty [21] by mapping the k-good path of length kn to a path from $(0, 0)$ to $((k-1)n+1, n)$ that does not touch the line $x = (k-1)y+1$. Then one obtains from (1.11) [21] that the number of such paths is given by $|L'(1, n; k-1)| = C_n^k$.

For $k = 2$, k-good paths are usually referred to as *Dyck paths*. It is well-known that they are counted by $\frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see [30, A000108] and [32]). Note that $(k+1)$-good paths are in one-to-one correspondence to *k-Catalan paths*, which are paths from $(0, 0)$ to $(n \cdot k, n)$ using only east-steps and north-steps, staying weakly below the line $y = x/k$.

We will now define our structure: Let $\mathcal{A}_n^k$ be the staircase in the $\mathbb{R}^2$ plane consisting of all points (x, y) such that $0 \leq x \leq n$ and $y \leq (k-1)\lceil n - x \rceil$. Figure 1 shows the staircases $\mathcal{A}_n^3$ for $n = 1, 2, 3$.

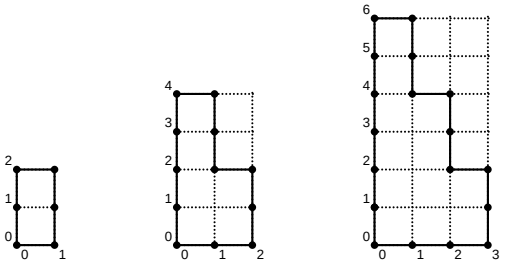


FIGURE 1. The staircases $\mathcal{A}_n^3$ for $n = 1, 2, 3$.

We refer to the set of points of the staircase $\mathcal{A}_n^k$ between the lines $y = j - 1$ and $y = j$ as the j-th *row* of $\mathcal{A}_n^k$ for $1 \leq j \leq (k-1) \cdot n$, and to the line $x = 0$ of the staircase $\mathcal{A}_n^k$ as the *border* of $\mathcal{A}_n^k$. Clearly, $\mathcal{A}_n^k$ has exactly $(k-1)n$ rows.

A *row-tiling* of the staircase $\mathcal{A}_n^k$ is a tiling in which each row of the staircase $\mathcal{A}_n^k$ is tiled with rectangular tiles of size $1 \times m$, $m \geq 1$, which we will call tiles of *size* or *length* m. Since we are mostly interested in the tiles of size $m \geq 2$, we will refer to them as *large* tiles. We will call a row which has only tiles of size 1 a *row without large tile*, and a row in which a tile fills the row completely (i.e., in row i, the tile is of size $n + 1 - \lceil \frac{i}{k-1} \rceil$) a *complete row*. Note that rows $(k-1)(n-1)+1$ to row $(k-1)n$ are always complete rows, since they only admit a single tile of size 1. A row-tiling is said to be *border* if it is a row-tiling that has at most one large tile in each row of $\mathcal{A}_n^k$, and the large tile is adjacent to the staircase border. The tile adjacent to the border (whether large or not) is called the *border tile*. A border row-tiling is said to be *heap* if the large tiles decrease in size from bottom to top, that is, if rows i and j have tiles of size p and q, respectively, then $i < j$ implies $q \leq p$.

If a row-tiling is both border and heap, then it is said to be a *BHR-tiling* (Border Heap Row-tiling). The sets of all border row-tilings of $\mathcal{A}_n^k$ is denoted by BR_n^k, and the set of *BHR*-tilings is denoted by BHR_n^k. Figure 2 shows the four border row-tilings of the staircase $\mathcal{A}_2^3$, of which all but the rightmost one are heap.

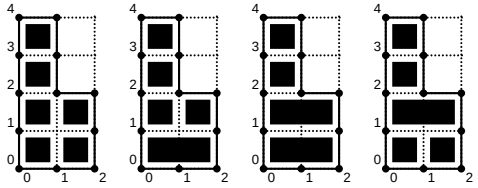


FIGURE 2. Border row-tilings of the staircase $\mathcal{A}_2^3$.

We are now ready to show a bijection between BHR_n^k and G_n^k and enumerate several statistics on the tilings. We first describe an algorithm to create a *tiling path* associated with a *BHR*-tiling, which will be used to create a k-good path associated with the tiling, and will be referred to in the proofs of the various bijections. The algorithm will be illustrated in Figure 3, which shows a *BHR*-tiling of $\mathcal{A}_3^3$, its tiling path, and the associated 3-good path and 3-ary tree. Before describing the path algorithm formally, we will give a geometric description. The tiling path essentially traces the right edges of the border tile in each row. However, to make this path into a k-good path, it needs to start at position $(0, (k-1)n)$, the top left corner, not the top right corner of the staircase. Also, the path needs to end at $(n, 0)$. Therefore, we shift the path one unit to the left (so that it now "shadows" the right end of the border tiles), and then add a horizontal segment at the end to obtain the proper end point.

Path Creation Algorithm. To create a tiling path associated with a *BHR*-tiling, follow these steps:

Step 1: Start at position $(0, (k-1)n)$.

Step 2: If the current path ends at position (i, j) and the border tile in row j ends at $x = \ell$, then extend the path horizontally from (i, j) to $(\ell-1, j)$, and then vertically to $(\ell-1, j-1)$. (Note that $\ell > i$, since the tiling is heap, and the path always stays to the left of the right end of the border tile.) Set $i := \ell - 1$ and $j := j - 1$. If $j > 0$, repeat Step 2.

Step 3: Complete the path by adding a horizontal segment from $(i, 0)$ to $(n, 0)$.

It is easy to see that each tiling creates an unique tiling path, which always starts at $(0, (k-1)n)$, ends at $(n, 0)$, and never crosses the line $y = (k-1)(n-x)$ (since the staircase A_n^k is contained in the half plane $y \leq (k-1)(n-x)+1$, and the path always stays one unit to the left of the corner points of the staircase). The path always starts with at least $k-1$ vertical steps (from $(0, (k-1)n)$ to $(0, (k-1)(n-1))$) since the top $k-1$ rows contain only a tile of size 1, and ends with at least one horizontal step (from $(n-1, 0)$ to $(n, 0)$), since the longest possible tile in the bottom row is of size n.

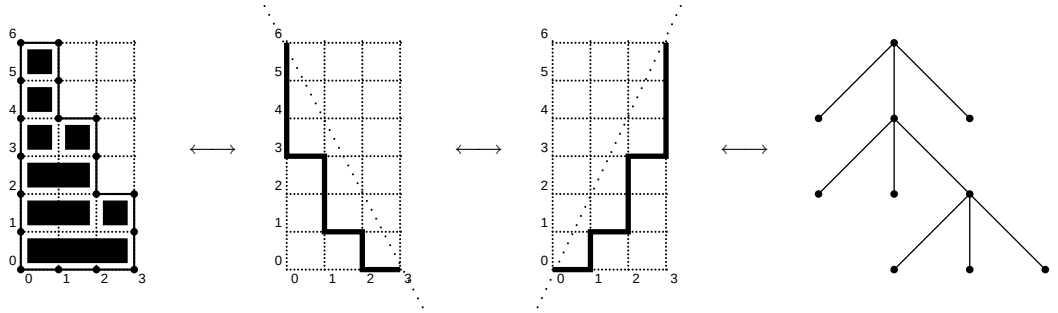


FIGURE 3. *BHR*-tiling with associated tiling path, k-good path, and k-ary tree.

Obviously, reflecting this tiling path on the vertical line $x = n/2$ results in a k-good path in G_n^k. Since both the reflection and the algorithmic translation are reversible, this gives a bijection between the tiling paths of *BHR*-tilings of A_n^k and k-good paths in G_n^k, leading to the following result.

Theorem 2.1. *There is a bijection between the set of BHR-tilings of A_n^k and the set of k-good paths G_n^k.*

In [14, 32], the bijections between the set of k-good paths and k-ary trees are given in a non-direct way. We describe a recursive tree creation algorithm which will give an easy direct bijection between these two sets. In particular, the labels given to the tree vertices will be used to create the k-good path.

Tree Creation Algorithm. To create the set of k-ary trees with n internal vertices, follow these steps:

- Start with the empty set. If $n > 0$, let $a_1 = 0$, and create the tree T_1 with root labeled v_1, having k leaves.
- For $1 \leq i \leq n$, let T_i be a tree with i internal leaves and associated sequence $(a_1, a_2, \ldots, a_i)$. Traverse T_i in preorder and label its leaves by $0, 1, \cdots, (k-1)i$ (these are all the leaves of T_i). For $a_{i+1} =$ one of $a_i, \ldots, (k-1)i$, create a set of trees T_{i+1} with associated sequences $(a_1, a_2, \ldots, a_i, a_{i+1})$ by adding a copy of T_1 at the leaf labeled a_{i+1}.
- Continue until $i = n$. The resulting set of trees T_n is the set of k-ary trees with n internal nodes.

Note that since $a_{i+1} \geq a_i$, the new subtrees are always added on the right of any existing subtree, so no duplicate trees are created. Furthermore, each k-ary tree can be produced in this manner. The creation of one particular tree in T_3, the one with associated sequence $(0, 1, 3)$, is shown in Figure 4.

The bijection between k-good path and k-ary trees is as follows: We read off the sequence of heights of the east steps of the path, and create the k-ary tree that has this sequence as labels, and vice versa.

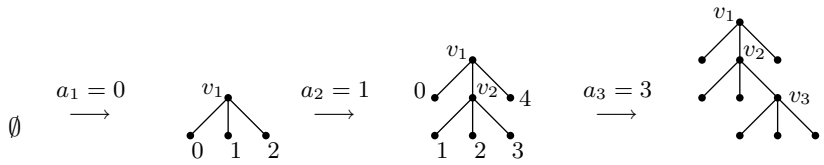


FIGURE 4. The tree creation algorithm.

For example, the height sequence for the k-good path in Figure 3 is $(0, 1, 3)$, which is the sequence for the tree created in Figure 4.

Theorem 2.2. *There is a bijection between k-good paths and k-ary trees, and therefore, $|BHR_n^k| = C_n^k$.*

More conveniently, we would like to rotate the k-good path counterclockwise around the origin until the line connecting the start and end points coincides with the x-axis, resulting (after an appropriate right shift) in a *k-ary path* of length kn, which is a lattice path from $(0, 0)$ to $(kn, 0)$ with up-steps $U = (1, \frac{1}{k-1})$ and down-steps $D = (1, -1)$, never falling below the x-axis. (Note that east steps are mapped to down steps, and north steps are mapped to up-steps.) The set of all k-ary paths of length kn is denoted by P_n^k, and the set of 3-ary paths of length 9 is shown as item 3 in Section 3. We say a step is at *level i* if the y-coordinate of its end point is at level i.

We generalize the well known first return decomposition [8], so that we can study the statistics of staircase tilings through the corresponding statistics of k-ary paths. Any k-ary path can be decomposed as $UP_1UP_2 \cdots UP_{k-1}DP_k$, where the last down-step of P_i (if it exists; $1 \le i \le k-1$) is the last down-step at level $\frac{i}{k-1}$ before D, which is the first down step at level 0. We call this the *first-last return decomposition*; an example for $k = 3$ is illustrated in Figure 5.

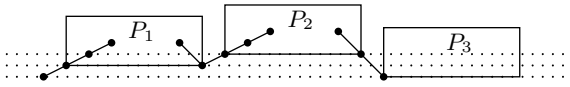


FIGURE 5. The first-last return decomposition of a 3-ary path.

We will now derive results on statistics on k-ary paths based on this first-last return decomposition that generalize some of the statistics known for Dyck paths [8, 20, 35]. For each statistic we obtain a recursive equation for the generating function, then apply the Lagrange Inversion Formula (see for example [40], Theorem 5.1.1), keeping y "fixed". We translate features of the k-ary path into characteristics of the staircase tiling.

Enumeration of k-ary paths according to the number of *DD*. In the k-ary path, a D occurs exactly when two adjacent border tiles in a *BHR*-tiling have unequal length. (We adjoin a row 0 with a tile of size $n + 1$ to account for the *DD*'s that are created by the last sequence of horizontal steps in the tiling path.) Specifically, if the difference in the size of the border tiles in rows i and $i - 1$ is $l \ge 1$, where $i = 1, \ldots, (k-1)n$, then we say a *change of size l* occurs. Note that a change of size l creates $l - 1$ occurrences of *DD* in the corresponding k-ary path. Let

$$d(T) = \sum_{i=1}^{(k-1)n-1} d_i, \qquad \text{where } d_i = \begin{cases} l-1 & \text{if a change of size } l \text{ occurs} \\ 0 & \text{otherwise} \end{cases},$$

i.e., $d(T)$ is the sum of all changes minus the number of changes. Thus, if $f(x, y)$ is the generating function for k-ary paths of length kn enumerated according to the number of *DD*, then

$$f = f(x, y) = \sum_{T \in BHR_n^k} x^n y^{d(T)} = \sum_{P \in P_n^k} x^n y^{\#DD \text{ steps in } P}.$$

By the first-last return decomposition, distinguishing whether P_{k-1} is empty or nonempty, we have

$$f = 1 + xf^{k-2}f + xyf^{k-2}(f-1)f.$$

Using the Lagrange inversion formula on the above recursive equation for $u = f - 1$ and treating y as a parameter, we obtain (after some algebraic simplification) the following result.

Theorem 2.3. *Let $f(x,y) = \sum_{P\in P_n^k} x^n y^{\#DD \text{ steps in } P}$. Then*

$$[x^n y^m]f(x,y) = \frac{1}{n}\binom{(k-1)n}{n-m-1}\binom{n}{m},$$

where $[x^n y^m]f(x,y)$ denotes as usual the coefficient of $x^n y^m$ in $f(x,y)$. In particular, the number of tilings T in BHR_n^k with $d(T) = m$ equals $\frac{1}{n}\binom{(k-1)n}{n-m-1}\binom{n}{m}$.

Setting $k = 3$ and $m = d(T) = 0$ in Theorem 2.3, we obtain that the number of tilings in BHR_n^3 that have changes of size at most one equals the n-th Catalan number, $\frac{1}{n+1}\binom{2n}{n}$.

Note that the sequence $[x^n y^m]f(x,y)$ for $k = 3$ occurs as A120986 in [30], and counts the number of ternary trees with n internal nodes and m middle children. The correspondence between these two combinatorial objects is easily seen from Figure 5. A *DD* in the 3-ary path occurs exactly if the middle part of the path, represented by the box labeled P_2, is nonempty. Since the tree creation algorithm is based on the height sequence, the bijection is clear - a *DD* occurs exactly when a middle child occurs and vice versa. For $k > 3$, the corresponding sequences are not in the On-line Encyclopedia. However, the correspondence observed for $k = 3$ works in general: if we label the children in a k-ary tree from left to right as $1, \dots, k$, the number of *DD*s corresponds to the number of children labeled $k - 1$.

Enumeration of k-ary paths according to the number of *UDU*. Note that each pair of adjacent rows with change of size one in a *BHR*-tiling creates an occurrence of *UDU* in the k-ary path by the bijection given in Theorem 2.1. Let $s(T)$ be the number of pairs of adjacent rows with change of size one in the tiling T. Thus, if f is the generating function for k-ary paths enumerated according to the number of *UDU*, then

$$f = f(x,y) = \sum_{T\in BHR_n^k} x^n y^{s(T)} = \sum_{P\in P_n^k} x^n y^{\#\ UDU \text{ steps in } P}.$$

Let $g = g(x,y)$ be the generating function of P_i $(1 \le i \le k-2)$ in the first-last return decomposition, i.e., g is the generating function for k-ary paths that are followed by an up-step U. Looking at four cases according to whether P_{k-1} and P_k are empty or non-empty, we obtain

$$f = 1 + x\,g^{k-2} + x\,g^{k-2}(f-1) + x\,y\,g^{k-2}(f-1) + xg^{k-2}(f-1)^2.$$

Using the same decomposition and the respective cases, but keeping in mind that the path is followed by an up-step U, we obtain this recursion for g:

$$g = 1 + x\,y\,g^{k-2} + x\,g^{k-2}(f-1) + x\,y\,g^{k-2}(g-1) + x\,g^{k-2}(f-1)(g-1).$$

Solving this system of equations in f and g for f gives $f = 1 + x\,f\left(1 + \frac{(f-1)(y+f-1)}{f}\right)^{k-1}$. We use once more the Lagrange inversion formula for $u = f-1$ and expand the resulting formula using the binomial theorem and the fact that $\left(\frac{1}{1+u}\right)^j = \sum_{s=0}^{\infty}(-1)^s\binom{j-1+s}{s}u^s$. After some algebraic simplification, we obtain the following result.

Theorem 2.4. *Let $f(x,y) = \sum_{P\in P_n^k} x^n y^{\#\ UDU \text{ steps in } P}$. Then*

$$[x^n y^m]f(x,y) = \frac{1}{n}\sum_{i=m}^{n-1}(-1)^{n-1-i}\binom{n(k-1)}{i}\binom{i}{m}\binom{n(k-1)+m-2-2i}{n-1-i}, \qquad \text{for } k \ge 3, \tag{2.1}$$

and

$$[x^n y^m] f(x,y) = \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} \binom{i}{m} \binom{i-m}{n-1-i} \qquad \text{for } k=2. \tag{2.2}$$

In particular, the number of tilings in BHR_n^k with m pairs of adjacent rows with change of size one is given by (2.1) *and* (2.2)*, respectively.*

Setting $k=2$ and $m=0$ in Theorem 2.4, we obtain that the number of tilings in BHR_n^2 with no adjacent rows with change of size one equals the n-th Motzkin number, $\frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{i}{n-1-i}$. This recovers the result shown in [35], namely that Dyck paths without UDUs are enumerated by the Motzkin numbers.

Enumeration of k-ary paths according to the number of UU. Note that adjacent rows with the same size border tile in a *BHR*-tiling create an occurrence of UU in the k-ary path. Let $t(T)$ be the number of adjacent rows with the same size border tile in a tiling T. Then we have

$$f = f(x,y) = \sum_{T \in BHR_n^k} x^n y^{t(T)} = \sum_{P \in P_n^k} x^n y^{\#UU \text{ steps in P}}.$$

Using the first-last return decomposition, and considering whether P_{k-1} is empty or non-empty, we have

$$f = 1 + xy^{k-2} f^{k-1} + xy^{k-1}(f-1) f^{k-1}.$$

Using the Lagrange inversion formula for $u = f-1$, followed by some algebraic simplification, we obtain the following result.

Theorem 2.5. *Let $f = f(x,y) = \sum_{P \in P_n^k} x^n y^{\#UU \text{ steps in } P}$. Then*

$$[x^n y^m] f(x,y) = \frac{1}{n} \binom{(k-1)n}{m+1} \binom{n}{(k-1)n-m}.$$

In particular, the number of tilings in BHR_n^k with exactly m pairs of adjacent rows having the same size border tile is given by $\frac{1}{n}\binom{(k-1)n}{m+1}\binom{n}{(k-1)n-m}$.

If $k=2$, the number of tilings in BHR_n^2 with m pairs of adjacent rows having the same size border tile is counted by the n-th Narayana number, $\frac{1}{n}\binom{n}{m}\binom{n}{m+1}$. This recovers a result by Deutsch [8] that the number of UUs in Dyck paths is counted by the Narayana numbers. Setting $k=3$ and $m=n$ in Theorem 2.5, we obtain that the number of tilings in BHR_n^3 with n pairs of adjacent rows having the same size border tile is given by the n-th Catalan number, $\frac{1}{n+1}\binom{2n}{n}$.

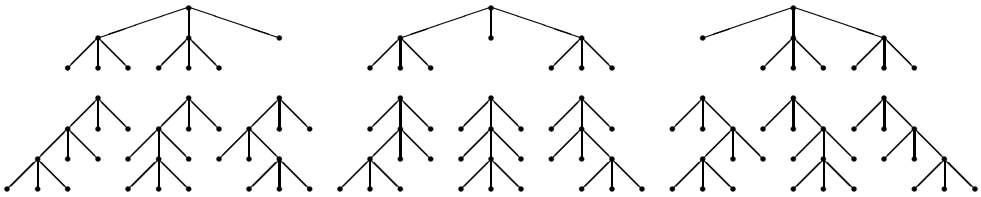
3. A Sampling of k-Catalan structures

In this section we provide a sampling of combinatorial objects known to be enumerated by the k-ary numbers and either give references or indicate bijections that show that they are indeed enumerated by the k-Catalan numbers. In making the selection, we have chosen some of the more interesting types from the larger list which can be found at www.math.haifa.ac.il/toufik/enum2005.html. Most of the bijections are straight forward generalizations of the respective Catalan bijection, so we provide hints, but do not give complete proofs. However, the proof for structure (9) is not so simple, and we give a detailed proof to show that the specific k-ary paths are enumerated by C_n^k.

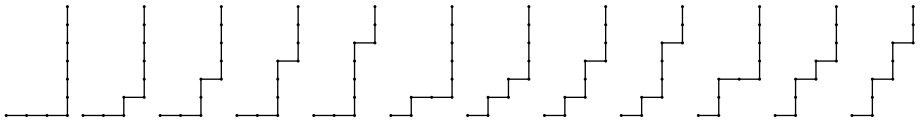
Rather than give very technical definitions for each of the structures below, we give illustrations of the set of objects for the case $k=n=3$, hoping that the pictures speak for themselves, except when the combinatorial object might not be widely known. Since these structures are generalizations of Catalan type structures given in [32, Ex. 6.19] and [33], we will also indicate Stanley's labels for the

respective Catalan object at the end of each description, where for example [(a)] refers to [32, Ex. 6.19 (a)]. Items labeled (a) through (nnn) appear in [32, Ex. 6.19], and items (ooo) through (e^6) appear in [33]. Note that the labels in [33] are changing as new objects are added to the list, so the labels given here may not remain accurate. The labels we have used are the ones of the version of October 30, 2005.

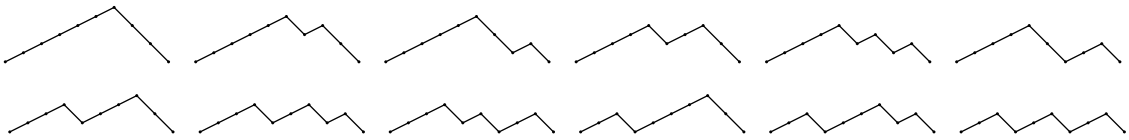
1. Plane k-ary trees with $kn+1$ vertices (or $(k-1)n+1$ endpoints) [(d)], (see [14]):



2. Lattice paths from $(0,0)$ to $(n,(k-1)n)$ with steps $(0,1)$ or $(1,0)$, never rising above the line $y=(k-1)x$, also called k-good paths [(h)], (see [14]):

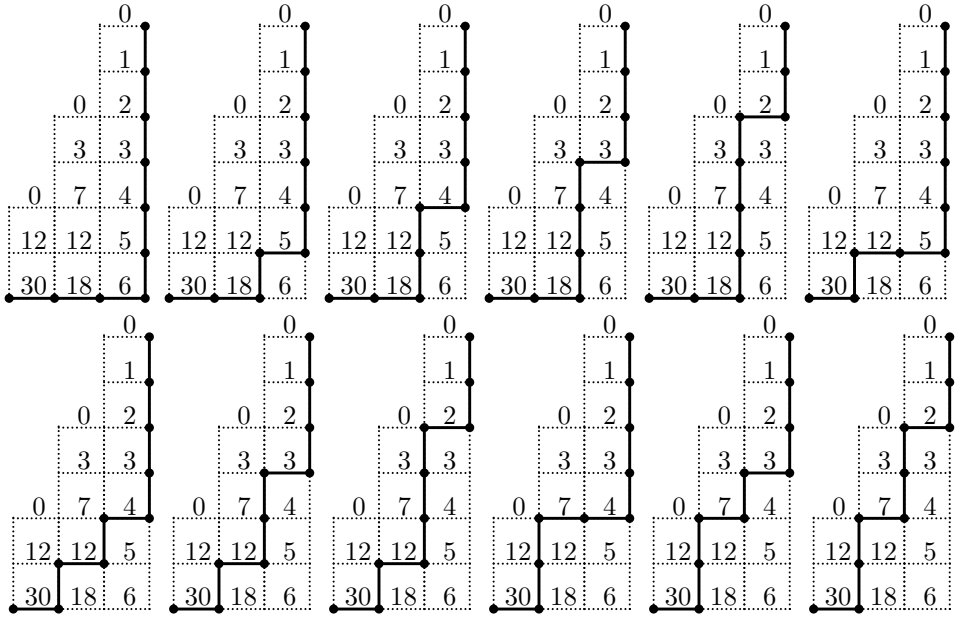


3. k-ary paths of length kn, i.e., lattice paths from $(0,0)$ to $(kn,0)$ with steps $(1,\frac{1}{k-1})$ and $(1,-1)$, never falling below the x-axis [(i)], (see [32, Proposition 6.2.1]):



4. The ranking sequences of k-good paths (see (2)) ([27], Section 4):

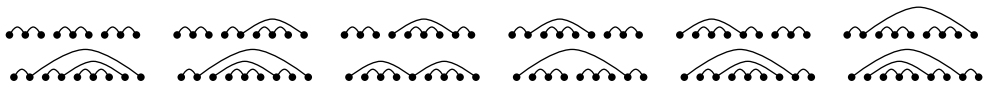
$6,18,30$	$5,18,30$	$4,18,30$	$3,18,30$	$2,18,30$	$5,12,30$
$4,12,30$	$3,12,30$	$2,12,30$	$4,7,30$	$3,7,30$	$2,7,30$



The ranking sequences are obtained as follows: the rightmost column is filled from top to bottom with the integers $0, 1, \ldots, n(k-1)$. Then the columns are filled from right to left by assigning a 0 as the topmost value in each column. The remaining values are computed as the sum of the value above and the value to the right of it. The ranking labels are the ones above the east steps of the path, read from right to left.

The following objects are new generalizations of combinatorial structures counted by the Catalan numbers. "New" indicates that we are not aware of a general proof that these structures are enumerated by C_n^k for any n and k. In some instances, the object described has been defined elsewhere (e.g. set-valued Young tableaux), but the enumeration proof was not given, or given only for specific values of n or k.

5. Ways of connecting kn points in the plane lying on a horizontal line by n nonintersecting (k-1)-arc sequences, where each (k-1)-arc sequence connects k of the points and lies above the points [(o)]:



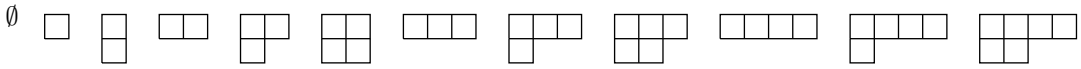
Hint: Reading the points on the line from left to right, create a k-ary path by appending a down step if a point is an end point of a (k-1)-arc sequence, otherwise appending an up step. For example, the first structure in (5) is mapped to *UUDUUDUUD* and the second structure in (5) is mapped to *UUDUUUUDD*. To create the $(k-1)$-arc sequences from a k-ary path P of length kn, let $i_1 < i_2 < \cdots < i_n$ be the sequence of the positions of the down steps D in the path P. Connect the vertex i_j, $j = 1, 2, \ldots, n$, with the closest $k-1$ non-connected vertices on its left side. For example, the 3-ary path *UUUUDUUDD* has the sequence $5 < 8 < 9$ of down steps, and maps to the sixth structure in (5). This gives a bijection between (3) and (5).

6. Column-strict plane partitions of shape $((k-1)(n-1), (k-1)(n-2), \cdots, k-1)$, such that each entry in the i-th row is equal to $n-i$ or $n-i+1$ [(yy)]:

3333	3332	3322	3333	3332	3322	3222	3333	3332	3322	3222	2222
22	22	22	21	21	21	21	11	11	11	11	11

Hint: Let b_i be the number of entries in row i that are equal to $n-i+1$ (so $b_n = 0$). Then sequences $b_n, b_{n-1}, \cdots, b_1$ obtained in this way are in bijection with the paths in (2) where b_i is the height of the i-th east step of the path.

7. Young diagrams that fit in the shape $((k-1)(n-1), (k-1)(n-2), \cdots, k-1)$ [(vv)]:



Hint: If $\lambda = (\lambda_1, \cdots, \lambda_{n-1}) \subseteq ((k-1)(n-1), (k-1)(n-2), \cdots, k-1)$, then the sequences $(0, \lambda_{n-1}, \cdots, \lambda_1)$ are in bijection with the paths in (2) where λ_i is the height of the i-th east step of the path.

For the following structure we provide a definition, as set-valued Young tableaux may not (yet) be widely known. Note that this definition is similar to the one given by Buch [5], except that he allows weakly increasing sets across rows, i.e., the definition by Buch is for semi-standard set-valued Young tableaux.

Definition 3.1. *A* standard set-valued Young tableau with shape $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and weights $w_{i,j}$ *is a tableau in which the cell at position* (i, j) *is filled with an integer set* $B_{i,j}$*, where*

- $|B_{i,j}| = w_{i,j}$ *and* $\bigcup B_{i,j} = \{1, \ldots, \ell\}$, *with* $\ell = \sum_{i,j} w_{i,j}$,
- $B_{i,j} \bigcap B_{i',j'} = \emptyset$ *unless* $i = i'$ *and* $j = j'$,
- $max(B_{i,j}) < min(B_{i+1,j})$ *and* $max(B_{i,j}) < min(B_{i,j+1})$, *for* $j < \lambda_i$

where $max(B_{i,j})$ $(min(B_{i,j}))$ *stands for the maximum (minimum) integer in* $B_{i,j}$. *If the weights across a row are the same, then we give the weights as* $(w_1, w_2, \ldots, w_r)$, *where* w_i *is the weight for row* i.

8. Standard set-valued Young tableaux of shape (n, n) with weight $(k-1, 1)$ (see [15] for $k = 3$) (or equivalently, of shape $(n, n-1)$ with weight $(k-1, 1)$) [(ww)]:

12 34 56	12 34 57	12 34 58	12 34 67	12 34 68	12 34 78
7 8 9	6 8 9	6 7 9	5 8 9	5 7 9	5 6 9

12 35 67	12 35 68	12 35 78	12 45 67	12 45 68	12 45 78
4 8 9	4 7 9	4 6 9	3 8 9	3 7 9	3 6 9

or

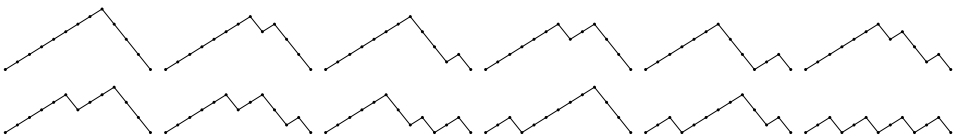
12 34 56	12 34 57	12 34 58	12 34 67	12 34 68	12 34 78
7 8	6 8	6 7	5 8	5 7	5 6

12 35 67	12 35 68	12 35 78	12 45 67	12 45 68	12 45 78
4 8	4 7	4 6	3 8	3 7	3 6

Hint: Given a standard set-valued Young tableau T of shape (n, n), define $a_1 a_2 \cdots a_{kn}$ by $a_i = U$ if i appears in row 1 of T, while $a_i = D$ if i appears in row 2. This sets up a bijection with (3), where $U = (1, \frac{1}{k-1})$ and $D = (1, -1)$. Note that the entries in the first row of the tableau give the positions of the up steps, and the second row gives the position of the down steps of the path.

For the last structure given here, the Catalan proof does not generalize, so we provide a detailed proof.

9. k-ary paths (as defined in (3)) from $(0, 0)$ to $(k(n+1), 0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$ [(k)]:



Proof: Let $g(x)$ be the generating function for the number of k-ary paths from $(0, 0)$ to $(kn, 0)$, i.e., lattice paths from $(0, 0)$ to $(kn, 0)$ with steps $(1, \frac{1}{k-1})$ and $(1, -1)$, never falling below the x-axis. Let $F_j(x)$ be the generating function for the number of k-ary paths of length kn with no peaks at height $\frac{i}{k-1}$ for all $i = k-1, k, \ldots, j$. Let such a path P be decomposed according to the the first-last return decomposition of k-ary paths (see Figure 5 for the case $k = 3$) as $P = UP_1UP_2 \ldots UP_{k-1}DP_0$. Then P does not have a peak at level $\frac{j}{k-1}$ if and only if P_i does not have a peak at level $\frac{j-i}{k-1}$. Therefore, in the generating function F_j each P_i contributes F_{j-i} for $i = 1, \ldots, j-(k-1)$, $P_{j-k+2}, \ldots, P_{k-2}$ each contribute a factor of g, P_{k-1} contributes $g-1$ (to avoid peaks at level 1) and P_0 contributes F_j. Solving for F_j, we obtain

$$F_j(x) = \frac{1}{1 - xF_{j-1}(x)F_{j-2}(x)\cdots F_{k-1}(x)g^{2k-3-j}(x)(g(x)-1)},$$

for all $j = k-1, k, \ldots, 2k-3$. By induction on i we can prove for all $i = 0, 1, 2, \ldots, k-2$, that

$$F_{k-1+i}(x) = \frac{1 - xg^{k-1-i}(x)(g(x)-1)\sum_{j=0}^{i-1} g^i(x)}{1 - xg^{k-2-i}(x)(g(x)-1)\sum_{j=0}^{i} g^i(x)},$$

and therefore,

$$\prod_{i=0}^{k-2} F_{k-1+i} = \frac{1}{1 - xg^0(g-1)(1+g+g^2+\cdots+g^{k-2})} = \frac{1}{1+x-xg^{k-1}}.$$

Now, let $h(x)$ be the generating function for the number of k-ary paths from $(0,0)$ to $(kn,0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$. Again, the first-last return decomposition of k-ary paths gives

$$h(x) = 1 + xF_{k-1}(x)F_k(x)\cdots F_{2k-3}(x)h(x).$$

Hence,

$$h(x) = \frac{1}{1 - x\prod_{i=0}^{k-2} F_{k-1+i}(x)} = \frac{1}{1 - \frac{x}{1+x-xg^{k-1}}} = 1 + \frac{x}{1-xg^{k-1}(x)}.$$

Since $g(x) = 1 + xg^k(x)$ (using the first-last return decomposition), we have that $\frac{g-1}{xg} = g^{k-1}$, and therefore,

$$h(x) = 1 + \frac{x}{1 - \frac{g-1}{g}} = 1 + \frac{xg}{g-g+1} = 1 + xg(x),$$

which shows that the number of k-ary paths from $(0,0)$ to $(k(n+1),0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$ equals the number of k-ary paths from (0,0) to $(kn,0)$, as requested. ■

4. Summary

We have provided a new structure enumerated by the k-Catalan numbers, new statistics on k-ary paths, as well as a direct bijection between k-good paths and k-ary trees. Using this bijection, we can translate the statistics on k-ary paths to statistics on staircase tilings. We also have given a sampling of combinatorial structures enumerated by the k-Catalan numbers and provide a more extensive, dynamic list on our website. We hope this website will become a repository for k-Catalan structures, and we invite the combinatorics community to submit additional structures enumerated by the k-Catalan numbers via email to the third author.

Acknowledgments The authors would like to thank the reviewers for their helpful comments on restructuring the article and the suggestion to start a website repository.

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