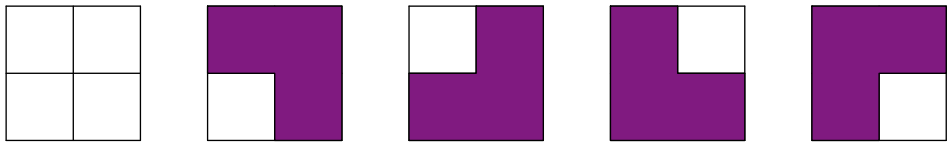
Tiling with Ls and Squares

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joint work with
Phyllis Chinn and Ralph Grimaldi

Many authors have looked at all kinds of tilings problems ... a **tiny** sampling:

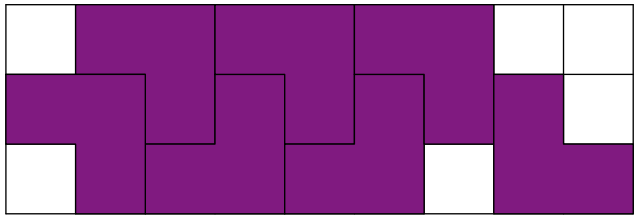
- E.O. Hare and P.Z. Chinn, Tiling with Cuisenaire rods, *Applications of Fibonacci numbers* **6** (1994), 165–171.
- S. Heubach, Tiling an m-by-n area with squares of size up to k-by-k with $m \leq 5$, *Congressus Numerantium* **140** (1999), 43-64.
- R. Hochberg and M. Reid, Tiling with notched cubes, *Discrete Math.* **214** (2000), 255–261.
- A. T. Benjamin and J.J. Quinn, *Proofs that Really Count*, MAA 2003

Things to come ...

- We look at $2\times n$ and $3\times n$ boards
- Count number of tilings $T_{m,n}$ of an $m\times n$ board
- Count number of Ls, $T^L_{m,n}$, and squares, $T^S_{m,n}$, in all such tilings
- $2\times n \rightsquigarrow$ explicit results, "easy case"
- $3\times n \rightsquigarrow$ different techniques needed

Definitions and basic ideas

- The generating function for a sequence $\{a_{m,n}\}_{n=0}^{\infty}$ is defined as $G_{a_m}(x) = \sum_{n=0}^{\infty} a_{m,n} x^n$.
- A basic block is a tiling that cannot be split vertically into smaller rectangular tilings. We denote the number of basic blocks of size $m \times k$ by $B_{m,k}$
- Any tiling is composed of a basic block followed by a smaller tiling.



$$\begin{aligned} T_{m,n} &= \sum_{k=1}^{n} B_{m,k} \cdot T_{m,n-k} \quad \text{for} \quad m,n \geq 1 \quad (*) \\ T_{m,0} &= 1 \quad \text{for} \quad m \geq 1 \end{aligned}$$

Recursion for $T_{m,n}$ is a convolution $\Rightarrow$ generating functions multiply. Multiplying (*) by x^n and summing over $n \geq 1$ we obtain

$$G_{T_m}(x) - 1 = G_{B_m}(x) G_{T_m}(x)$$

$$\Rightarrow G_{T_m}(x) = \frac{1}{1 - G_{B_m}(x)}$$

Connection between the various quantities

- Looking at total area gives

$$m \cdot n \cdot T_{m,n} = 3T^L_{m,n} + T^S_{m,n}$$

$\Rightarrow$ need to count only squares or Ls

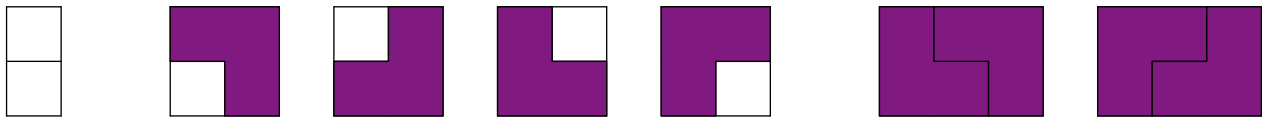
- Counting Squares

$$T^S_{m,n} = \underbrace{\sum_{k=1}^{n} B_{m,k} \cdot T^S_{m,n-k}}_{\text{squares from tilings}} + \underbrace{\sum_{k=1}^{n} B^S_{m,k} \cdot T_{m,n-k}}_{\text{squares from basic blocks}} \qquad (**)$$

- Corresponding formula holds for Ls

Tiling $2\times n$ boards

- Basic blocks of size $2\times k$



- Recursion $T_{2,n} = T_{2,n-1} + 4 \cdot T_{2,n-2} + 2 \cdot T_{2,n-3}$
- Characteristic equation $x^3 - x^2 - 4x - 2$ has roots -1, $1 + \sqrt{3}$, and $1 - \sqrt{3}$

Theorem: The number of tilings of size $2\times n$ with L-shaped tiles and squares for $n \geq 0$ is given by

$$T_{2,n} = (-1)^n + (1/\sqrt{3})(1+\sqrt{3})^n + (-1/\sqrt{3})(1-\sqrt{3})^n,$$

with generating function

$$G_{T_{2,n}}(x) = 1/(1 - x - 4x^2 - 2x^3).$$

The values for $T_{2,n}$ for $n = 0, \ldots, 20$ are given by {1, 1, 5, 11, 33, 87, 241, 655, 1793, 4895, 13377, 36543, 99841, 272767, 745217, 2035967, 5562369, 15196671, 41518081, 113429503, 309895169}

Counting Ls and Squares

We give formulas for $T^L_{2,n}$ and $T^S_{2,n}$ in three forms: combinatorial, explicit and generating function.

Combinatorial set-up:

- Tiling consists of sequence of basic blocks
- $s = \#$ of basic blocks of width 1, $d = \#$ of basic blocks of width 2, $t = \#$ of basic blocks of width 3.
- $s + 2d + 3t = n \Rightarrow$

  $\#$ of basic blocks $\ell = s + d + t = n - d - 2t$

Theorem: The number of squares and L-shaped tiles for tilings of the $2\times n$ board are given in combinatorial form by

$$T^S_{2,n} = \sum_{t=0}^{n/3} \sum_{d=0}^{(n-3t)/2} \binom{\ell}{t}\binom{\ell-t}{d}(2s+d)4^d 2^t$$

$$T^L_{2,n} = \sum_{t=0}^{n/3} \sum_{d=0}^{(n-3t)/2} \binom{\ell}{t}\binom{\ell-t}{d}(d+2t)4^d 2^t,$$

where $s = n - 3t - 2d$.

Explicitly:

$$\begin{aligned} T^S_{2,n} &= (2n-12)(-1)^n \\ &\quad +\frac{2}{3}((9-5\sqrt{3})(1+\sqrt{3})^n+(9+5\sqrt{3})(1-\sqrt{3})^n) \\ &\quad +\frac{n}{\sqrt{3}}((\sqrt{3}-1)(1+\sqrt{3})^n+(\sqrt{3}+1)(1-\sqrt{3})^n). \end{aligned}$$

and

$$\begin{aligned} T^L_{2,n} &= 4(-1)^n-\frac{2}{9}((9-5\sqrt{3})(1+\sqrt{3})^n+(9+5\sqrt{3})(1-\sqrt{3})^n \\ &\quad -\frac{n}{3}((1-\sqrt{3})(1+\sqrt{3})^n+(1+\sqrt{3})(1-\sqrt{3})^n). \end{aligned}$$

The respective generating functions are

$$G_{T_2^S}(x) = \frac{2x(1+2x)}{(1+x)^2(1-2x-2x^2)^2}$$

and

$$G_{T_2^L}(x) = \frac{4x^2}{(1+x)(1-2x-2x^2)^2}.$$

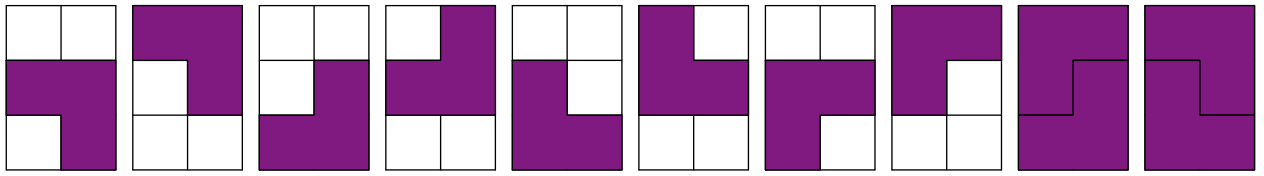
Proof (Outline): From (**) we get (for Ls)

$$\begin{aligned} T^L_{2,n} &= T^L_{2,n-1} + 4 \cdot T^L_{2,n-2} + 2 \cdot T^L_{2,n-3} \\ &\quad +4 \cdot T_{2,n-2} + 4 \cdot T_{2,n-3} \end{aligned}$$

- Homogeneous part $\leftrightarrow T_{2,n}$; for inhomogeneous part substitute result for $T_{2,n} \Rightarrow$ linear comb. of $(1+\sqrt{3})$ and $(1-\sqrt{3})$
- Solve general solution for $T^L_{2,n}$ using initial conditions, then use area formula for $T^S_{2,n}$
- Generating functions easy - (**) consists of two convolutions, and $G_{B^S_2}(x) = 2x + 4x^2$ and $G_{B^L_2}(x) = 4x^2 + 4x^3$.

Tiling $3\times n$ boards

- One basic block of size $3\times1 \leftrightarrow$ all squares
- Eight basic blocks of size 3×2



- Basic blocks of bigger size ????? $\rightsquigarrow$ recursive algorithm to create basic blocks of size $3\times(k+1)$ from those of size $3\times k$
- Look at the last "column" and refer to each 1×1 area of the tiling as a cell

Basic Block Creation (BBC) Algorithm:

- *Type I*: The last column has one cell covered by a square and two (adjacent) cells covered by an L.
- *Type II*: The last column has two non-adjacent cells covered with squares.
- *Type III*: The last column has two adjacent cells covered with squares.

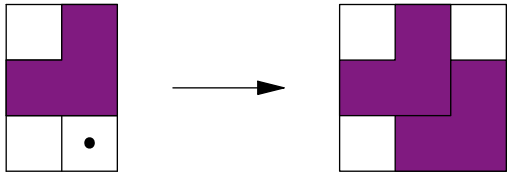
Other possibilities for last column:

- All cells are covered with squares (only for $k = 1$)
- All cells are covered with L tiles (only for $k = 2$)

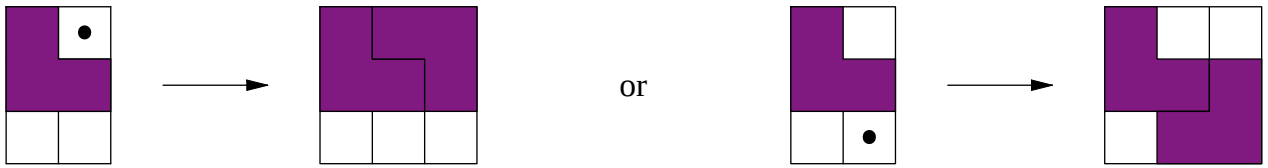
Neither of these basic blocks can be extended.

Extensions

- Each Type I extension produces one Type I basic block

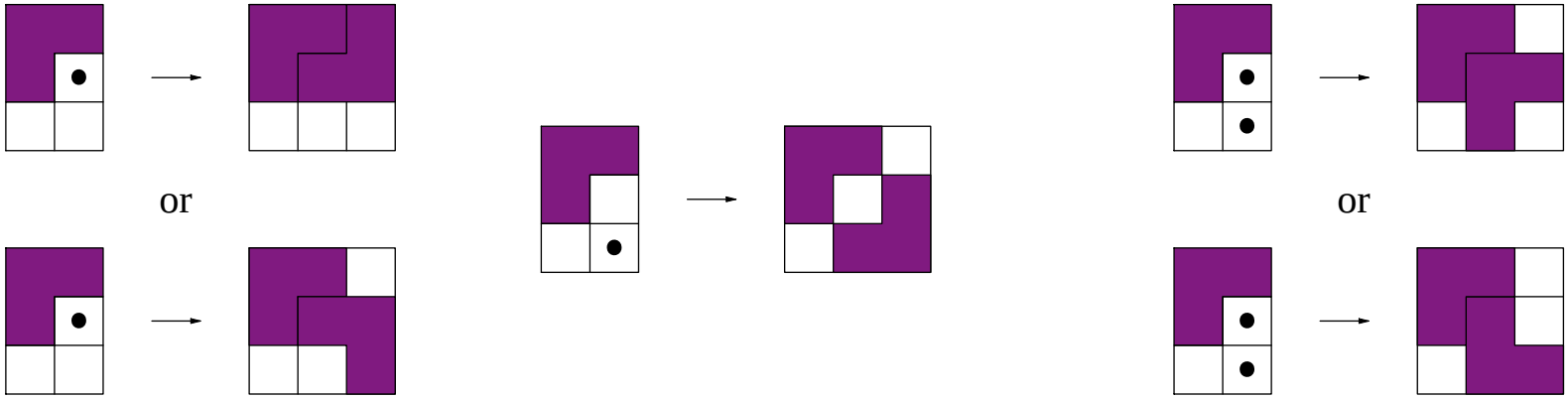


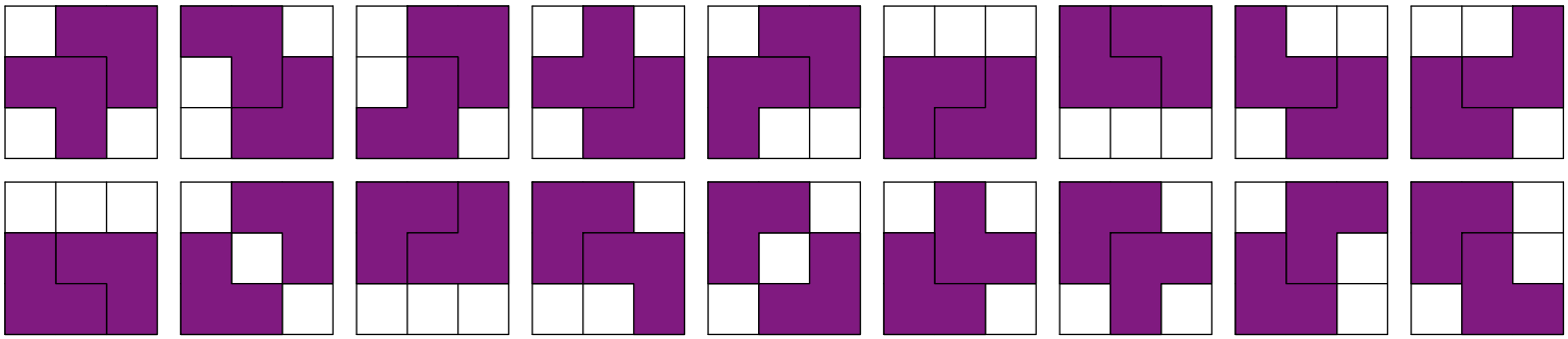
- Each Type II extension produces two Type I basic blocks

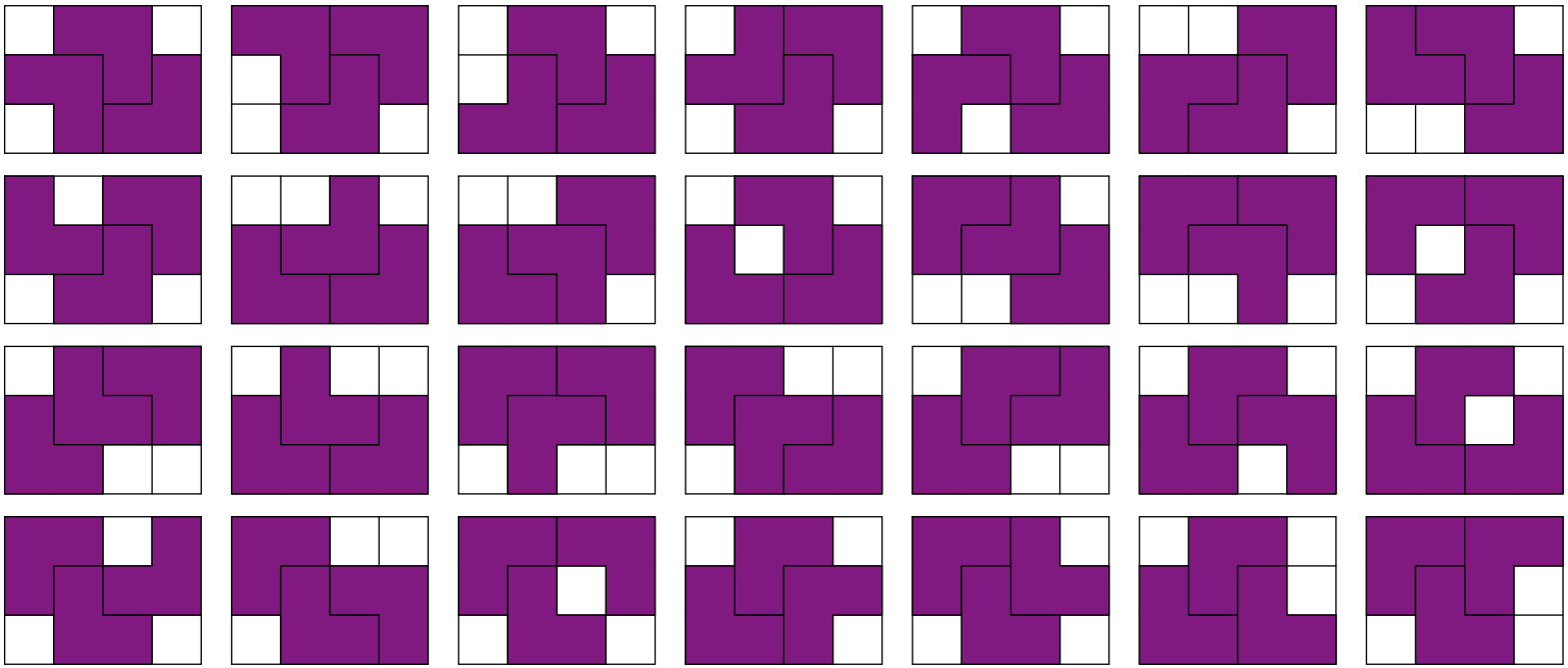


Type III Extensions

Each Type III extension produces three Type I, one Type II, and one Type III basic blocks.







Theorem: The number of basic blocks is given by

$$B_{3,1} = 1, B_{3,2} = 10, \text{ and } B_{3,k} = 10k - 12 \quad \text{for } k \geq 3.$$

The generating functions for the number of basic blocks and the number of tilings are given by

$$G_{B_3}(x) = \frac{x + 8x^2 - x^3 + 2x^4}{(1-x)^2}$$

and

$$G_{T_3}(x) = \frac{(1-x)^2}{1 - 3x - 7x^2 + x^3 - 2x^4}.$$

The values for $\{T_{3,n}\}_{n=0}^{15} = \{1, 1, 11, 39, 195, 849, 3895, 17511, 79339, 358397, 1620843, 7326991, 33127155, 149766353, 677103839, 3061202815\}$.

Proof (Outline):

- For $k \geq 3$, $B_{3,k} = b_{k,I} + b_{k,II} + b_{k,III}$
- Initial conditions $b_{2,I} = 4$, $b_{2,II} = 2$, and $b_{2,III} = 2$
- Extension algorithm gives recursions $\rightsquigarrow$ solve

$$\begin{aligned} b_{k+1,III} &= b_{k,III} (= \ldots = b_{2,III} = 2) \\ b_{k+1,II} &= b_{k,III} = 2 \\ b_{k+1,I} &= b_{k,I} + 2b_{k,II} + 3b_{k,III} = b_{k,I} + 10. \end{aligned}$$

- Gf for $B_{3,k}$ from explicit formula; gf for $T_{3,k}$ from convolution

Theorem: The generating functions for the number of squares and L-shaped tiles in all tilings of the $3\times n$ board are given by

$$G_{T_3^S}(x) = \frac{3x(1-x)^2(1+6x+3x^2)}{(1-3x-7x^2+x^3-2x^4)^2}$$

and

$$G_{T_3^L}(x) = \frac{4x^2(3-3x+3x^2-4x^3+x^4)}{(1-3x-7x^2+x^3-2x^4)^2}.$$

Proof (Outline):

- Note that each basic block (for $k \geq 3$) has exactly 3 squares - proof follows from the extensions
- Compute gf for $B^S_{3,n}$ from gf of $B_{3,n}$ with adjustments for initial discrepancy
- Gf for $T^S_{3,n}$ from

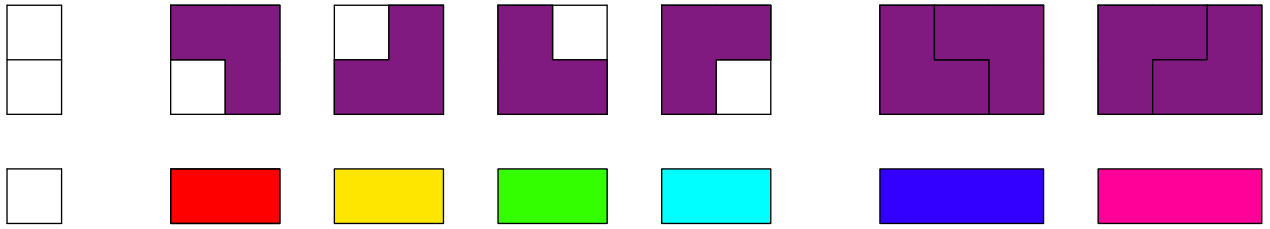
$$T^S_{m,n} = \underbrace{\sum_{k=1}^{n} B_{m,k} \cdot T^S_{m,n-k}}_{\text{squares from tilings}} + \underbrace{\sum_{k=1}^{n} B^S_{m,k} \cdot T_{m,n-k}}_{\text{squares from basic blocks}} \qquad (**)$$

- Gf for $T^L_{3,n}$ from area formula

Connection to colored $1\times n$ boards

- Any finite recursion with the proper initial condition can be interpreted as counting the tilings of a $1\times n$ board with colored tiles of size $1\times k$
- Coefficients in the recursion indicate the number of tiles of the respective size
- Each basic block of a given width is mapped to a different color

$$T_{2,n} = T_{2,n-1} + 4 \cdot T_{2,n-2} + 2 \cdot T_{2,n-3}$$



Future Research

- Case $m > 3$??? $\rightsquigarrow$ there are more types for the extension algorithm
  - Is there a general structure for the types?
  - Is the a general structure for the types and number of each type that result from extension of a given type?
- Tiling $2 \times n$ and $3 \times n$ bracelets

Thanks for Listening

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